Analysis and Synthesis

REPRINT

On Scientific Method -
Based on a Study by Bernhard Riemann

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Abstract - This article deals with the foundations of analysis and synthesis as scientific methods, and especially with the requirements for the successful application of these methods. Although analysis and synthesis always go hand in hand – they complement one another – there are important situations in which one method can be regarded as more suitable than the other. This concerns the question of which method is most appropriate as the primary point of departure for the study of a given system or object of scientific inquiry.

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Section I Introduction

A. *Background*

The terms *analysis* and *synthesis* come from (classical) Greek and mean literally "to loosen up" and "to put together" respectively. These terms are used within most modern scientific disciplines -- from mathematics and logic to economy and psychology -- to denote similar investigative procedures. In general, *analysis* is defined as the procedure by which we break down an intellectual or substantial whole into parts or components. *Synthesis* is defined as the opposite procedure: to combine separate elements or components in order to form a coherent whole.

Careless interpretation of these definitions has sometimes led to quite misleading statements -- for instance, that synthesis is "good" because it creates wholes, whereas analysis is "bad" because it reduces wholes to alienated parts. According to this view, the analytic method is regarded as belonging to an outdated, reductionist tradition in science, while synthesis is seen as leading the "new way" to a holistic perspective.[1]

Quite aside from the fact that it is the synthetic method which, historically, is associated with a reductionist approach to scientific inquiry, such interpretations arise from a fundamental misunderstanding of the relationship between these two methods.

Analysis and synthesis, as scientific methods, always go hand in hand; they complement one another. Every synthesis is built upon the results of a preceding analysis, and every analysis requires a subsequent synthesis in order to verify and correct its results. In this context, to regard one method as being inherently better than the other is meaningless.

There are, however, important situations in which one method can be regarded as more *suitable* than the other. This concerns the question of which method is most appropriate as the primary method or chief point of departure for the study of a given system or object of scientific inquiry.

This article deals with the foundations of analysis and synthesis as scientific methods, and especially with the requirements for the successful application of these methods. The article is based upon a short, brilliant methodological study by the 19th century mathematician Bernhard Riemann. The study in question is a fragment of Riemann's last (unfinished) work, "The Mechanism of the Ear", written in 1866.

In order to gain a better understanding of Riemann's intentions in writing "The Mechanism of the Ear", this introduction continues with a biographical sketch and a discussion of the issues and controversies surrounding his last work. In Section II, the first 13 paragraphs of Riemann's manuscript are presented, with detailed comments to each paragraph.

It may well be that many readers already have a good working knowledge of the analytic and synthetic methods. I believe, however, that all of us -- practicing scientists and systems analysts as well as teachers and students -- have much to learn from Riemann's discussion concerning the requirements for the applicability of these two methods.

B. *Bernhard Riemann*[2]

Bernhard Riemann was born in the little German kingdom of Hanover in 1826. He was the son of a rural priest and was apparently expected to follow in his fathers footsteps by studying theology. However, his talent for mathematics was discovered at an early age and his teachers at the German Gymnasium arranged for him to study the classics of science. When he entered the University at the age of 20, his knowledge and general scientific background was far in advance of the curriculum offered at the time.

Riemann entered the University at Göttingen in 1846, initially in order to study theology. In 1887 he moved to Berlin to study mathematics under Jacobi and Dirichlet, returning to Göttingen in 1849. In 1851 he put forward his doctoral dissertation and, in 1854, became Privatdozent -- that is, an unpaid lecturer. Finally, in 1859, he became full professor of mathematics at Göttingen University.

Riemann was famous during his own lifetime; he was recognized as a genius by the greatest mathematicians of the day. However, he remained "unemployed" -- i.e. had no formal position at the university and lacked regular income -- until he succeeded Dirichlet as professor in 1859. Years of economic hardship and overwork evidently so weakened his health that he died, 39 years old, in 1866.

Probably more than any other single person, Riemann influenced the course of modern mathematics. During his short, intense career he published only a handful of papers, but all of them were important. Several have opened up new, productive areas of mathematical study and a few have been nothing less than revolutionary.

Riemann is known primarily for the so-called Cauchy-Riemann equations, Riemann surfaces and Riemannian geometry, Riemann's differential equation, the Riemann integral, Riemann's zeta-function and the Riemann hypothesis. However, the basis for most of his discoveries seems to rest upon his fundamental development of the theory of complex functions. The methods he developed in this area led him to other discoveries in analysis, geometry, number theory and even hydrodynamics -- subjects which today are thought of as belonging to more or less separate areas of study.[3]

Riemann's fundamental ideas on geometry were presented in his famous Inaugural Address of 1854, "On the Hypotheses which lie at the Foundations of Geometry".[4] This short, non-technical work is undoubtedly one of the highlights in the history of science. In it he analyses the basic assumptions which underlie geometry and develops unified principles not only for the classification of all then existing forms of geometry, but also for the creation of any number of new types of space. He later developed the basic analytical tools, which Einstein would subsequently use, in his theory of general relativity.

Riemann's work on aerodynamic equations ("On the Propagation of Plainer Air Waves of Finite Amplitude") resulted in new developments in the theory of hyperbolic partial differential equations and in the mathematical description of a shock wave -- a phenomenon that was demonstrated experimentally only some 50 years later. As late as the 1940's, his classical treatment of aerodynamic shock waves remained unique.[5]

Riemann has also left mathematics with one of its outstanding, unsolved problems: the Riemann hypothesis. In a short article, often referred to as "the most important paper ever published in number theory"[6], Riemann tackled the classical problem of the number of prime numbers F(x) that are less than a given number x. By applying complex analysis to the problem he arrived at a hypothesis which has revealed much about the distribution of prime numbers and which provides a remarkable interaction between (mathematical) analysis and arithmetic.

This represents only a small portion of Riemann's most important work. However, the methods he developed in order to facilitate his discoveries, and the panoramic way in which he approached mathematical problems in general, have proved to be no less important. Much of his work is associated with methodological discussions, in which he attempts to display the very thought process which underlie his discoveries. The clear and simple way in which he presents innovative mathematical ideas -- often without so much as writing down an equation -- is still read with great admiration.

C. *The Mechanism of the Ear*[7]

During the last years of his life -- spent mainly in Italy for the dryer climate -- Riemann worked with various aspects of natural philosophy, including an attempt to formulate a unified theory of light and gravity and the development of a general theory of knowledge.

He began his study of the mechanics of the ear in response to the publication of Hermann Helmholtz' thesis *Die Lehre von den Tonempfindungen als psysiologische Grundlage für die Teorie der Musik* (On the Sensations of Tone as a Physiological Basis for the Theory of Music, 1863).

Helmholtz was a leading proponent of the reductionist approach to physiology. Essentially, this approach is based upon the assumption that the study of biology (and therefore physiology) can be reduced to methods based upon classical (Newtonian) mechanics. It is upon this foundation that Helmholtz developed a theory to describe the mechanism of the human ear. He did this by making careful anatomical studies and by applying the laws of acoustics, as they were understood at the time. He then attempted to apply this theory to the principles of musical composition, harmony and counterpoint.

There is much in Helmholtz' study of the physiology of the ear which has been of great and lasting value. Especially his careful anatomical investigations led to a greatly improved empirical foundation for the study of the ear's mechanism. However, the *theory* he formulated concerning this mechanism -- and its premature application to the theory of musical composition -- was in many ways flawed.

One of the central features of Helmholtz' theory concerned the role and function of the middle ear. Helmholtz contended that when a complex tone is transmitted from the eardrum to the cochlea (inner ear), by way of the ossicles (the three small bones in the middle ear), it is transmitted not as a geometrical whole, but in the form of a set of primary tones. The middle ear "breaks down" the complex tone into its primary components, each component consisting of a given frequency and intensity. Each primary tone then stimulates a specific area or resonator in the cochlea, and each resonator then transmits a signal to the brain by way of a specific nerve. This is what Helmholtz called his "theory of specific energies".

One may -- in retrospect -- smile at Helmholtz' overly mechanistic view of the workings of the middle ear. However, we must not forget that he had at his disposal instruments and experimental techniques that were a far cry from those that are available today. It must have been extremely difficult, if not impossible, to examine the mechanical properties of a living, functioning ear in any detail.

There were, of course, problems with Helmholtz' theory.[8] Among other things, it could not explain the production of harmonics and combination tones. Helmholtz thus formulated a number of *ad hoc* hypotheses and explanations in order to make his theory complete. For example, he claimed that the middle ear always *distorts* incoming sound waves -- the "clicking" of the ossicles -- in a way that produces harmonics and combination tones.

Although acknowledged as an important physicist, as well as being renowned for his meticulous anatomical studies, Helmholtz' "theory of specific energies" divided German physiology into rival camps. Why, of all people, would Bernhard Riemann -- the greatest mathematician of his age -- devote his time to the enterprise of developing an alternative theory?

The answer to this question would seem to concern an issue of method. A hint of Riemann's intentions are given in Henle's forward to "The Mechanism of the Ear", published in Henle's and Pfeuffer's *Zeitschrift für rationelle Medicin* less than a year after Riemann's death.

> "... the publication of this fragment is undoubtedly justified -- given the stature of the author and the value of his assertions -- as an exemplary case of the methodological treatment of the subject."[9]

However, in Riemann's own introduction, "On the Method which is Applicable to the Study of the Physiology of the Finer Sense Organs", it is clear that his intentions go beyond simply presenting a better method for the study of this special subject. He uses this specific "case study" in order to discuss the wider applications of the analytic and the synthetic methods for scientific investigation in general. It is this introductory discussion, which includes an *analysis of the analytic method itself*, which I consider to be the main point of his study.

Before we begin with this methodological discussion, however, it is interesting to see how Riemann -- who openly admits that his knowledge of the anatomy of the ear derives mainly from Helmholtz' “excellent book on the subject”[10] -- approaches the matter of developing an alternative to Helmholtz' theory.

Firstly, Riemann does not begin his investigation form the point of view of the ear's *anatomy,* but from what the ear actually *accomplishes,* i.e. its sensitivity, fidelity and discrimination. This must be determined, with all possible precision, empirically -- i.e. by way of observation and experiment.

He then asks: what task does the ear have -- what problem must it solve -- in order to accomplish what it does. It is the formulation and analysis of this **task** which is the starting point and primary focus of his investigation.

As for the anatomy of the ear, Riemann applies the approach advanced by his colleges and personal friends Ernst Weber and Gustav Fechner.[11] In this approach, the structure and role of any individual component of an organ can only be determined on the basis of its relationship to the function of the organ as a whole. One must not, at least at the outset, assume any specific functional relationships between individual components.

Finally, Riemann applies the principle of "projective invariance" as concerns the transmission of sound waves in the ear. He argues that some physical quantity inherent in the geometry of the wave pattern must remain invariant in its transmission from the eardrum to the cochlea -- if we are to explain what the ear actually accomplishes. He identified *timbre* -- a term that is usually referred to as the "quality of sound", but which is quantifiable -- as such an invariant.

Even in fragmented form, Riemann's manuscript contains a number of important conclusions. Firstly, he rejects Helmholtz' theory that the middle ear only transmits primary tones to the cochlea. To the contrary, the ear must "transmit to the fluid of the inner ear the variation in the air pressure at every moment at a constant ratio of amplification. ... we consider it justified to assume that the timbre curve is altered only very slightly by the transmission".[12]

Riemann's ear perceives the "quality of sound" -- its timbre curve -- as a geometric whole or gestalt. This also means that Riemann rejects Helmholtz' *ad hoc* hypothesis that the middle ear "distorts" the sound wave in order to produce harmonics and combination tones.

In this context, modern studies in physiological acoustics have shown Riemann -- on the whole -- to be correct.[13] But Riemann's main point is not simply to reject Helmholtz' *ad hoc* explanations; he is aiming at something greater than this. In order to really understand the special qualities of the ear -- i.e. those properties that the organ *must,* in some way or another, have in order to accomplish what it does -- we may need to develop new methods and even new scientific principles that can "give rise to advances in our knowledge of the laws of nature".

For Riemann, one of the more intriguing problems to be understood is the ear's ability to perceive sound waves, the energy levels and physical displacements of which are so small that they cannot even be measured directly.[14] Modern quantitative measurements of the human ear's sensitivity reveal that the sensory receptors of the cochlea can detect motions of atomic dimensions; and that we can perceive musical tones in which the energy transmission to the eardrum is of the order of 10^{-18} Joules![15]

D. *Riemann's Methodological Introduction to "The Mechanism of the Ear"*

As implied in the preceding discussion, the main issue in question in "The Mechanism of the Ear" is one of scientific method. However, Riemann goes beyond simply presenting an example of a "method which is applicable to the study of the physiology of the finer sense organs". He uses this example in order to address the wider implications of analysis and synthesis as investigative techniques, and to discuss the role of these methods in scientific discovery in general.

This he does in the first thirteen paragraphs of his introduction to "The Mechanism of the Ear". It is these thirteen paragraphs, which are presented in Section II -- in English translation and with an extended commentary.

The aim of my commentary is both to help the reader in interpreting Riemann's extremely compact text (if such help be needed), and to discuss the broader scientific and methodological implications of his arguments. My main purpose is to present an analysis of the analytical method itself, as this is put forward by one of the great creative minds in the history of science.

In my comments I take the liberty of employing the term "system" -- a term that Riemann does not use. However, the systems concept, if applied properly, is central to a discussion of the analytic and synthetic methods. Naturally, one must be careful here. In "The Mechanism of the Ear", Riemann is treating one special type of system; a system with its own distinctive characteristics as concerns its origins and experimental accessibility.

In very general terms, a system is any (circumscribed) object which consists of a number of "parts" or "components" which, in some way or another, work together in order to produce an overall effect or behavior. As immediately can be seen, this concept is so general and all-encompassing, that any attempt to define it in both a complete and logically consistent manner would probably be futile. We can only concede to the obvious: that just about everything in the world would seem to be some sort of "system".

However, there is a fundamental aspect of the systems concept which is central here. Regardless of any specific definition, we observe that the concept always distinguishes between (at least) two different levels of abstraction, or systems levels: the system as a functioning unit and the system as a set of interacting parts.

We regard a system as a primary unit when we treat it as a black box and ask about its overall behavior -- i.e. what it *does* or *accomplishes.* For example, we may submit our black box to various inputs and observe the resulting outputs.

As a set of parts or components (which somehow work together to produce the system's overall behavior) we can examine the system's *construction* -- i.e. its internal structure and processes.

Although there are other aspects of the systems concept which are ignored here, it is this distinction between system levels -- between the behavior of the system as a whole and the specific relationships between its parts -- which is fundamental to the concept. The idea of a system would be meaningless without this distinction.

These two system levels make up two foundations from which we can gain knowledge about a system. Furthermore, these levels are associated with different types of possible knowledge, and we can imagine the case (which in fact is the typical case) in which clear and reliable knowledge can only be gained concerning one of these levels. We can thus ask: to what degree is clear and reliable knowledge about these two systems levels accessible to us *in the case of any given system*?

Of course, the distinction between "clear and reliable" vs. "unclear and unreliable" knowledge is purely relative; and the degree to which we can ascribe knowledge on this basis is an empirical matter. However, as we shall see, the choice of a suitable method for the study of a given system depends, to a large extent, on the type of knowledge that is empirically accessible to us, as concerns these two cited systems levels.

Note on translation

Translating Riemann is indeed a challenge. His texts constitute hair-raising examples of 19th century academic German, with convoluted grammatical constructions and seemingly incomprehensible internal self-references. Fortunately, "The Mechanism of the Ear" is one of his more accessible writings.

The translation appearing here is based upon an English translation by Gallager, Cherry and Sigurdson[16]. Their translation is well balanced and captures much of Riemann's style. On the basis of my own Swedish translation, however, I have taken the liberty of changing a number of words and expressions which I believe better represent Riemann's intentions. These adjustments in no way alter the content of the original English translation.

However, it is not only Riemann's spectacular use of the German language which makes his texts so challenging. When one has finally broken through the purely formal, linguistic hinder, one finds -- to one's great surprise -- that Riemann in fact expresses himself quite clearly, and in a pedagogical style so simple and straight forward that it easily beguiles. Take care! This is elementary scientific method presented from a deceptively advanced standpoint.

Section II The Mechanism of the Ear (§ 1 - 13)[17] - with comments

Riemann's text is indented, presented in **bold text** and numbered by paragraph. I have emphasized certain key words by underlining and added explanatory phrases in brackets [ ].

On the Method Applicable for the Study of the Physiology of the Finer Sense Organs

§ 1 For the study of the physiology of a sense organ, there are -- aside from the universal laws of nature -- two necessary, special foundations: one psychophysical, i.e. the empirical determination of what the organ accomplishes; the other anatomical, i.e. the investigation of its construction.

For the study of any system -- including, for instance, the ear -- we have at our disposal three foundations upon which we can rely in order to gain knowledge about the system: one general foundation and two special foundations.

The first, *general foundation* consists of our knowledge of the laws of nature; knowledge which has been developed through centuries of scientific activity. This takes the form of scientific "laws" and principles, and systematic, empirically validated relationships. The laws of (theoretical) mechanics are an outstanding example of such acquired knowledge.

This knowledge is not absolute; it does not represent the final truth. We can, however, trust such knowledge as an approximate truth within its area of application and use it as a stepping-stone to new scientific discoveries.

The two *special foundations* we have at our disposal pertain to the system itself. The first of these concerns what the system performs or accomplishes. This we gain knowledge about by way of observation and experiment. Here we treat the system as a black box: how does it behave in different situations and how does it react to different external stimuli.

As concerns determining what the ear, as a system, accomplishes, Riemann uses the term "psychophysical". What the ear performs *for us*, is, so to speak, in our own heads. This idea of the psychophysical continuity of the finer sense organs originated with Riemann's friend and colleague, physiologist Ernst Weber. Weber pointed out that, in dealing with the ear or the eye, it is difficult to distinguish between physical sensation and "psychic" perception. Only under special circumstances -- for instance, in the case of strong, low frequency sonic waves -- do we actually "feel" sound waves on the eardrum. Normally, we simply "hear" sound as a psychophysical phenomenon. It is as though the ear is an extension of the brain itself -- a tentacle to the outside world.

The second special foundation for the study of any system is the system's *construction*. We gain knowledge of this by way of a detailed study of the system's internal structure and processes; and especially the "working relationships" between its constituent parts.

§ 2 **Accordingly, there are two possible ways of gaining knowledge about the organ's functions. Either we can proceed from its construction, and from there seek to determine the laws of the mutual interaction of its parts as well as its response to external stimuli; or we can begin with what the organ accomplishes and then attempt to account for this.**

§ 3 **By the first route we infer effects from given causes, whereas by the second route we seek causes of given effects. Following Newton and Herbart, we can call the first route synthetic, and the second analytic.**

Here Riemann uses the classical definition of the concepts of analysis and synthesis, which is expressed in the form of *cause* and *effect.* We can easily see how the cause-effect relationship is translated into a relationship between the two cited "special foundations" (which in essence are the two systems levels described in Section I).

A system's internal processes -- i.e. the interactions between its parts -- are regarded as the *cause* of what the system, as a unit, performs. What the system performs is thus the *effect.* From these very relationships we can immediately recognize the requirements for the application of the analytic and synthetic methods.

The synthetic approach -- i.e. *to infer effects on the basis of given causes* -- is therefore appropriate when the laws and principles governing a system's internal processes are known, but when we lack a detailed picture of how the system behaves as a whole.

For example, we do not have a very good understanding of the long-term dynamics of galactic systems, nor even of our own solar system. This is because we cannot observe these objects for the thousands or even millions of years which would be needed in order to map their overall behavior.

However, we do know something about the principles, which govern these dynamics, i.e. gravitational interaction between the stars and planets respectively. We can therefore apply a synthetic procedure in order to simulate the gross dynamics of these objects. In practice, this is done with the use of computer models which calculate the interaction of system parts over long, simulated time periods.

The analytical approach -- *drawing conclusions about causes on the basis of effects* -- is appropriate when a system's overall behavior is known, but when we do not have clear or certain knowledge about the system's internal processes or the principles governing these.

For example, we cannot directly observe the earth's internal structure and dynamics in order to establish the cause of continental drift. On the other hand, our knowledge about continental drift -- as an effect of this internal dynamic -- gives us a basis on which to infer these causes by drawing conclusions about the underlying processes.

There are, of course, many systems for which we have -- for all practical purposes -- clear and certain knowledge concerning both causes and effects: we understand both how the system behaves as a whole (or what it accomplishes) and how its internal parts and processes work in order to cause this behavior. Such systems can be called transparent.

On the other hand, there are a great many systems for which we neither have a clear and certain conception of how they behave as a whole, nor fully understand the principles at work which cause that behavior. Two examples of such systems are the human brain and (sadly) the national economy.

In defining the classical concepts of analysis and synthesis, it is interesting to note that Riemann names both Isaac Newton and Johan Fredrich Herbart as historical references. These two figures -- as Riemann well knew -- represent quite different traditions or schools of scientific thought.

Riemann, following Gauss and Herbart, was a leading representative of the so-called continental tradition which, to a large extent, rests upon the influence of G.W. Leibniz (1645-1715). Leibniz was among the first to define analysis and synthesis as modern methodological concepts:

> "*Synthesis* ... is the process in which we begin from principles and [proceed to] build up theorems and problems, ... while *analysis* is the process in which we begin with a given conclusion or proposed problem and seek the principles by which we may demonstrate the conclusion or solve the problem."[18]

Johan Fredrich Herbart (1776-1841), often referred to as the father of modern pedagogy, was professor of philosophy at the university of Göttingen. Brought up on Kantian philosophy, he eventually broke with his teacher, Fichte, and developed a philosophical-pedagogic program based upon the work of Leibniz. This meant, among other things, that Herbart opposed Newton's and Kant's assumption of the absolute nature of space and time.

Issac Newton (1642-1727) represented a competing tradition in science, the so-called British-empiricist school. Newton also formulated the principles of analysis and synthesis (for example in "Opticks", 1704) which, on the whole, are equivalent to Leibniz'.

Then, as now, Newton was raised to almost God-like status, but this did not prevent Riemann from being critical. In "On the Hypotheses which lie at the Foundations of Geometry" (1854), he points out Newton's unfounded assumption concerning the absolute nature of space -- a question which, 140 years earlier, resulted in a heated debate in the famous controversy between Leibniz and Newton.[19]

In "The Mechanism of the Ear", Riemann is critical of Newton's (misleading) assertion: "Hypotheses non fingo" Indeed, for Riemann it is the very process of hypothesis-formulation which is the basis, and very motor for scientific discovery.

At this point (§ 1-3), Riemann has explained the primary differences between analysis and synthesis as scientific methods. What remains is a discussion of how we go about applying these methods.

§4 **"The Synthetic Route. The first route is that which is most familiar to the anatomist. Since the anatomist is involved with the investigation of the individual components of the organ, he feels obliged to inquire of each part, what influence it might have upon the activity of the organ. This route could, with equal success, be taken with regard to the physiology of the sense organs -- as it has been with the physiology of the organs of locomotion -- provided that the physical characteristics of the individual parts of the [sense] organ could indeed be determined. But the determination of these characteristics from observation of microscopic objects is always more or less uncertain, and is, moreover, highly imprecise."**

Riemann refers here to the manner in which the "anatomist" approaches his subject. To begin from the point of view of the ear's anatomy, and then to draw theoretical conclusions from this about how the organ functions as a whole is, by definition, to apply the synthetic method.

It is by this method that the science of anatomy has successfully explained (at least on the macroscopic level) the workings of the organs of locomotion – e.g. arms and legs. Here it is a question of leverage, tension and torque etc., all of which can be investigated with good relative precision and within the framework of known theoretical-mechanical laws.

Riemann points out that the physiology of the ear could just as well be approached by this method -- *provided that the properties of the ear's individual parts could indeed be determined with sufficient precision*. But this requirement, i.e. the precise determination of the properties of these microscopic components, was quite impossible during the 19th century, and is still a great challenge today.

If the synthetic method is, by definition, based upon this very requirement, then the question arises: what happens when we apply this method to the study of a system which does not meet the requirements for this procedure's application.

§ 5 **"In order to complete such an inquiry, one is therefore compelled to resort to analogy or teleology, which unavoidably involves extreme arbitrariness. For this reason, the synthetic approach to the physiology of the sense organs leads to results that are seldom correct and, in any event, never very certain."**

In order to compensate for the fact that we lack sufficiently precise knowledge about how the components of the system actually function and interact, we find it necessary to "complement" our lack of knowledge by formulating explanations based upon something other than the anatomical "facts". We can do this by comparing and likening certain of the system's components with other mechanical objects or processes with which we are already familiar (analogy); or we can attribute to these components a purpose or utility vis à vis other components (teleology). In this way, *we force* the system's internal properties to conform to specific conceptions that are familiar to us.

With this we risk introducing into our explanation quite arbitrary conceptions, thus leading to results which either miss the point completely or, at best, make it difficult to determine if we are, at all, on the right track.

§ 6 **The Analytic Route. By the second route we seek to account for what the organ accomplishes. This undertaking can be broken down into three parts.**

1. The search for an hypothesis which is sufficient to explain what the organ accomplishes.

2. The investigation of the extent to which this explanation is a necessary one.

3. Comparison with experience in order to verify or correct [the explanation].

Here Riemann lists three phases in an analytic investigation. These phases are not, however, to be regarded as simple, algorithmic steps to be taken in sequence. In practice, the procedure involves interaction and mutual justification between the three phases.

One begins by "the search for an hypothesis which is sufficient to explain what the organ accomplishes".

§7 **I. We must, as it were, [conceptually] reinvent the organ; and, insofar as we consider what the organ accomplishes to be its <u>purpose</u>, we must also consider its formation as the <u>means</u> to that purpose. However, this purpose cannot be based upon presumption, but rather is given by experience; and if we disregard how the organ was actually [physiologically] produced, we need not at all bring into play the concept of final cause.**

In order to formulate an hypothesis about how a system works -- a system whose internal properties cannot be determined with any certainty -- we must identify something about the system which is "analyzable", but which is not directly dependent upon knowledge about its construction.

In order to do this, we must recreate the system *in principle*, i.e. build up a conceptual model of the system which, in some abstract way or another, contains the "sense of its functioning", so to speak. It is this conceptual model which is the object of analysis.

But how -- on what foundation -- do we "reinvent" or "recreate" the system? It is here that we turn the synthetic method on it head: instead of using cause to explain effect, we use effect to explain (something about) cause. If we cannot, to any sufficient degree, understand the system's construction in order to explain what it accomplishes, then we ought to begin with what the system accomplishes in order to secure a (sound) theoretical framework within which to explain its construction.

If we -- in a methodological sense -- consider what the ear accomplishes to be it "purpose" *for us*, then we can regard its construction as a *means to achieve that purpose*. And since we know that the internal properties of the ear in fact cause the effect that they do, then we have, as a matter of fact, a very definite sort of knowledge about these internal properties, namely: they must be sufficient to cause what the ear accomplishes.

This may, at first glance, seem like a circular argument -- self-evident and trivial. In fact, it contains nothing more than what is already implicit in the very definition of analysis. The crucial methodological point is: It is what the ear accomplishes, psychophysically, which is the key to understanding its construction -- not the other way around.

Furthermore, if the "purpose" of the ear is to perform what it in fact does perform, then we would do well to consider that it does this in its own unique way. It is naive of us to ascribe *ad hoc* mechanical principles to this "unbelievably sensitive instrument".

The use of the term "purpose" may seem somewhat provocative here. Its use is often looked upon with great suspicion -- at least in the context of the natural sciences. Such suspicion is of course warranted in the case of statements like: "God or Nature created the ear in order to ...". However, Riemann is very careful to point out that while "purpose", in this context, is a methodological concept, it is not something that is simply open to speculation, but is an empirical matter.

What Riemann wants at this point is a concept of what the ear does *in principle*, without any preconceived notions about *how* the organ actually, physiologically, carries out its task.

§ 8 "[Ultimately,] in order to explain what the organ actually accomplishes, we look to its construction. In our search for this explanation, however, we must <u>first analyze the organ's task</u> [i.e. the problem it must solve]. This will result in a series of secondary tasks [or problems], and only after we have become convinced that [all of] these must be solved, do we then look to the organ's construction in order to infer the <u>manner</u> in which they are solved."

To recap: In order to gain a complete understanding of a system's workings, we must eventually examine its construction. But we cannot begin here, and attempt to draw conclusions about how the system functions, if sufficient knowledge concerning its internal properties is not available to us. We must, instead, invert the process: we need some kind of theoretical framework within which we can first *draw conclusions about the system's internal properties*. And in order to do this, we must analyze the problem that the system must solve in order for it to accomplish what it does.

It is common to distinguish between two main types of analysis: *analysis of composition* (or of structure) and *analysis of function*.

In an analysis of composition, one (actually or conceptually) breaks down a system into its structural parts or components. It is this process which -- when used indiscriminately -- is rightly targeted as "reductionist". But this form of analysis is trivial: It is, in fact, precisely what the "anatomist" commits just before he begins to apply the synthetic procedure. It is for this reason that the synthetic procedure -- when applied in the way that Helmholtz applied it -- is historically associated with a reductionist approach to scientific inquiry.

In an analysis of function, one breaks down a system on the basis of identified functional processes or activities, which the system must carry out in order to perform what it does. For example, a (living) animal must have the capacity to take up and circulate energy; it must somehow be able to perceive its surroundings; and it must have some method of reproduction. The actual anatomical forms associated with these functions, however, may be quite different in different types of animals.

The analysis of a system's overall task lies on a different plane altogether. This task can be formulated as a problem, which the system -- in this case the ear -- must solve. Since this is to be formulated on the basis of what the ear actually accomplishes, and also within the framework of known natural laws, then we must be able to analyze this problem -- in principle.

It is thus the ear's **task** -- i.e. the main problem that the organ must solve -- which is to be analyzed. It is important here to understand that this has nothing to do with the ear's construction per se. At this point we are quite indifferent as to the actual physiological properties of the organ. Until we have analyzed the ear's principle task, we must refrain from speculating about the physiological manner in which this task is solved.

The analysis proceeds in the following way: the main task or problem that the ear must solve is broken down into a number of secondary tasks or problems, the solutions of which, in turn, are necessary for the solution of the main task. We continue this process, constructing a network or hierarchy of subordinate tasks, all of which must be solved in order that the originally formulated main task be solved. We stop the analysis when we notice that we are no longer formulating tasks to be solved, but have begun to formulate possible **means** by which they may be solved.
A task-analysis of this sort is no simple matter. It demands both sound empirical knowledge about a system's actual performance, and basic insight into the principles and "natural laws" that may be relevant to that performance.

When one has succeeded in mapping out this network or hierarchy of tasks, which contains all the conditions for the system to achieve the effect that it does, then one has formulated an hypothesis which is sufficient to account for the organ.

§ 9 **"II. Once we have arrived at a conception that is sufficient to explain the organ, we cannot fail to inquire about the extent to which this explanation is a necessary one. We must carefully distinguish between those assumptions that are unconditional -- or necessary by virtue of incontestable laws of nature -- and those classes of conception that could just as well be replaced by others. We thereby sort out all completely arbitrary, tacked-on [*ad hoc*] notions. Only in this way can we eliminate the detrimental consequences of the use of analogy in our search. This also makes it considerably easier to test our explanation by reference to experience (i.e. by framing questions to be answered)."**

A *sufficient explanation* for a phenomenon implies the following: if all the conditions contained in the explanation are in fact satisfied, then the phenomenon must exist and function according to the explanation. This means that a sufficient explanation for a given phenomenon is a *possible explanation* but not necessarily the correct explanation: there may be other explanations, involving other conditions, which also suffice to explain the phenomenon.

Let us say that we have constructed a hierarchy of conditions -- in the form of tasks or problems -- that must be satisfied (solved) if the system in question is to achieve the (known) effect that it does. These conditions are sufficient if the following is true: if the system **in fact** solves all these problems, then it must produce the desired effect. We have, so to speak, proved the possibility of the system, but we have not proved that it necessarily must solve all these particular problems.

A *necessary explanation* consists of conditions that must always be satisfied if the phenomenon is to exist at all. A necessary explanation need not be sufficient, but it is essential and "unconditional". Our next step, therefore, must be to work our way from an explanation that is merely sufficient, to an explanation that is necessary -- even if this means ending up with an explanation that is no longer complete.

We must "carefully distinguish" between those conceptions about the ear's functioning which are unconditional -- which always must be the case -- and those features of our explanation which are merely possible, but which could be replaced by other (just-as-possible) solutions. We do this in order to guard ourselves against purely arbitrary, *ad hoc* notions which we may have imported into our explanation, as the result of having utilized analogy in the formulation of our original hypothesis.

If we succeed in identifying and sorting out the necessary conditions contained in our explanation -- those conditions which must be the case "by virtue of incontestable laws of nature" -- then we have a secure foundation upon which we can formulate questions that can be tested on the basis of known scientific principles.

§ 10 "III. To test our explanation by reference to experience, we can in part draw upon what we have concluded about what the organ accomplishes, and in part upon what that explanation <u>presupposes as to the physical characteristics of the organ's constituent parts</u>. As for what the organ accomplishes, this is extremely difficult to precisely compare with experience -- and for the most part we must confine our theory-testing to the question of whether the theory is contradicted by experimental results or observations. In contrast, conclusions about the physical characteristics of the constituent parts can have universal scope, and can give rise to advances in our knowledge of the laws of nature -- as was the case, for example, with Euler's efforts to account for achromatism of the eye."

When we have come to the point where we can begin to test our explanation empirically, we again have two foundations at our disposal: what the system performs or accomplishes, and its construction.

In the case of the ear, it is difficult to draw detailed conclusions about our explanation, based upon the organ's psychophysical performance. While we may be able to determine with some certainty what the ear accomplishes psychophysically, this can hardly be determined with any precision. For example, in determining whether we can hear a very weak tone, it is difficult to give any other answer than "yes" or "no". We can, on the other hand, check to see that our explanation does not contradict empirical knowledge.

What our explanation implies about the ear's physical construction, however, can lead us to more spectacular results. The very reason that the analytical method is applied in this case, and not the synthetic method, is because we lack detailed and certain knowledge about the internal properties of the organ. The whole point of the procedure is to build up a conceptual framework within which we can draw certain necessary conclusions about these properties. By analyzing the ear's "task" -- i.e. the technical problems that it must, in principle, solve -- we can begin to draw conclusions about what our conceptual model implies or "*presupposes as to the physical characteristics of the organ's constituent parts*".

The conclusions that we draw in this context may open up new avenues of approach, and may even lead us to new perspectives concerning the very physical laws and principles that we have been applying. Riemann gives the example of Euler's study of the eye's achromatism, which eventually led to a solution for constructing an achromatic lens.

* * *

At this point, Riemann concludes his discussion of the analytic method and begins his summation.

§ 11 "These two opposed investigative approaches, we might add, only correspond *a potiori* to the designations "synthetic" and "analytic". Purely synthetic and purely analytic research, when taken in the precise sense of these terms, is an impossibility. Every synthesis rests upon the results of a preceding analysis, and every analysis requires a subsequent synthesis in order that it may be confirmed or corrected with reference to experience. With the former, synthetic procedure, the universal laws of motion are simply the result of a previous, assumed analysis."

Riemann begins by pointing out that it is not a question of the absolute priority of one method over the other. Although analysis and synthesis, each taken in the abstract are "two opposed investigative approaches", they are in fact complementary methods and always presuppose one another. In other words, we can consider a given investigative approach as being analytic or synthetic only on the basis of which of these procedures is the primary or chief point of departure.

That these two methods in actuality presuppose one another is perhaps easier to see in the case where analysis is taken to be the dominant method. We apply the analytic procedure in order to create a conceptual framework within which we can draw conclusions about the means by which a system solves its tasks. Indeed, we do this with the express purpose of establishing a solid foundation from which we can carry out a subsequent synthesis. This synthesis, in turn, acts to verify the conceptual model as an explanation. The process is an iterative one.

The fact that the synthetic approach always rests upon a preceding analysis may be more difficult to appreciate -- at least until we go back to the very beginning of Riemann's discussion. There he identifies three foundations for the study of the ear. Two of these foundations spring from the ear itself: what the ear accomplishes and its physical construction. The third foundation, which Riemann names only in passing, is the "universal laws of nature" and especially the "general laws of motion" (theoretical mechanics).

But our knowledge of the "laws of nature" is not simply given by God or Nature. These laws are intellectual constructs, which have been developed through the centuries by means of precisely the same process which Riemann is describing in the case of the study of the ear. That is, by way of a continual interaction between analytic and synthetic procedures. The "universal laws of motion" are themselves the result of analytic procedures applied to hypotheses about space, time and causality.

Although analysis and synthesis make up two sides of the same coin, the fact remains that every given object of scientific investigation is, in a sense, unique. Each has its own special properties which determine the type and degree of empirical knowledge that is available to us. And it is this issue of the "availability of knowledge" which to a large extent governs our choice of the most suitable method to be applied to a given system.

§ 12 "The first, chiefly synthetic procedure should therefore not be utilized for a theory of the finer sense organs, because the requirements for the applicability of this procedure are only incompletely met, and any attempted completion of these requirements by way of analogy and teleology will remain completely arbitrary."

In general, a system, the internal properties of which cannot be determined with any accuracy or certainty, does not fulfill the requirements for the application of the synthetic method. This is given by the very definition of the synthetic procedure: to infer effects from known or given causes. If one attempts to remedy this problem by tacking-on to the system *ad hoc* internal properties, then we risk introducing completely arbitrary notions into our explanation.

Such arbitrary notions can be introduced when we make analogies -- based upon our knowledge of other mechanisms -- about how specific components of a system function; or when we "teleologically" attribute to these components specific utilities vis à vis other components.

However, Riemann does not completely reject the use of analogy and teleology, if only these are used in a proper and cautious way:

§ 13 "In the case of the second, chiefly analytic approach, we still cannot entirely dispense with teleology and analogy, but can avoid their arbitrary use if we:

1) confine the application of teleology to inquiry into the *means by which the organ accomplishes its tasks*, but without raising the question of the *utility of its individual components*; and

2) do not -- as Newton proposes -- completely reject the use of analogy (the "poetry of hypothesis"), but rather afterwards emphasize the conditions that *must* be met to account for what the organ accomplishes, and discard any notions that are not essential to the explanation, but that have arisen solely through the use of analogy.

There is, then, a very important difference between the analytic and synthetic methods as concerns the use of analogy and teleology. This has to do with the fact that these two methods "intrude upon" a system at two different levels. The analytic approach -- in the sense that Riemann is using it -- takes as its point of departure the system as a primary unit, from the point of view of what is accomplishes as a non-reducible whole; whereas the synthetic method begins with the system's physical composition -- a system of interactions.

With the analytic method, teleological reasoning can be used in a methodologically sound manner, because we are concerned with the principles by which a system performs its task, or solves its problem. And since we already know what the system does, then we also know that its physical components must, somehow, act in a way that is sufficient to accomplish this task. As long as we concentrate on the question of how the system as a whole relates to its task -- and not on how its specific parts relate to one another -- then we may use teleological reasoning as an aid in our investigation.

The use of teleological arguments to explain how specific parts of a system function vis à vis one another is a different matter. In this context, attributing a purpose or utility to a specific component -- when this is precisely what we have not been able to establish, with any certainty, in our "anatomical" observation of the system -- is to risk attributing to the system arbitrary properties.

This argument may seem self-evident and trivial; it conveys nothing more than what is already given in the definitions of analysis and synthesis. This does not, however, diminish its importance. What Riemann seems to be saying is: if we keep in mind what analysis and synthesis are actually about, then the whole matter *is* self-evident. We cannot simply ascribe functional roles to specific components of a system, if this is precisely what is in question.

The same reasoning concerning teleology also applies to the issue of analogy -- although here the problem is somewhat more complex. Analogy, in this context, has to do with some sort of associative thinking which is, at once, free and disciplined. Analogy gives us a formal association between seemingly different types of phenomena. This association is based upon some underlying principle or pattern that both phenomena share, but which goes beyond any specific (physical) structure or process. Riemann calls analogy "the poetry of hypothesis", i.e. a medium of free but disciplined creation.

Both the place of analogy in scientific method, and the question of the creative process involved in "hypothesis formulation", have been discussed in science and philosophy for at least two thousand years.[20] As concerns analogy, there seems to be two, seemingly contradictory messages to be had. On the one hand, many of history's greatest scientific minds -- e.g. Leibniz, Gauss, Poincaré, Einstein, Weyl and Gödel -- have maintained that analogy is the very motor of scientific discovery. On the other hand, we are often warned about the dangers of making comparisons by analogy: we can easily go too far, and endow phenomena with functional properties that they simply do not have.

There is, however, no real contradiction here – as Riemann points out. Analogy is the means by which we can look at a system in a new way – "reinvent" it, so to speak – on the basis of the principles and laws which underlie its functioning. Analogy is best used to formulate hypotheses about such fundamental principles and the lawful conditions that a system must satisfy in order to perform what is does; not about the **specific manner** in which its components function vis à vis one another.

If we keep to this principle, then we have nothing to fear from the scientific use of analogy.

Notes

1. An example of such a misleading interpretation is to be found in R. Ackoff, *Redesigning the Future - A Systems Approach to Societal Problems* (New York: John Wiley & Sons, 1974) p. 8f:

   > "*Machine Age* thinking was *analytical* and based on the doctrines of *reductionism* and *mechanism*".
   >
   > "*Analytical thinking* is a natural complement to the doctrine of reductionism. It is the mental process by which anything to be examined, hence understood, is broken down into parts."

2. This biographical abstract is based upon three main sources: Lewy's Introduction to B. Riemann, *Gesammelte Mathematische Werke* (Dover, New York, 1953); L. Steen (ed.) *Mathematics Today* (New York, 1985); D. Struik, *A Concise History of Mathematics* (New York, 1967).

3. Steen, *op. cit.*. Riemann is credited with fundamental discoveries in four different areas of modern mathematics: the theory of complex functions, number theory, geometry, and methods in partial differential equations.

4. B. Riemann, *op. cit.*, 1953, pp. 272-287.

5. J. von Neumann, "Discussion of the Existence and Uniqueness or Multiplicity of Solutions of the Aerodynamical Equations", J. in von Neumann, *Collected Works*, vol. 6 (New York, 1963) pp. 384-356. Mathematician von Neumann pointed out, in 1951, that Riemann's classical investigation of aerodynamic equations was still unique.

6. L. Steen, *op. cit.*, chapter on Number Theory.

7. This section is based mainly on Gallager, R: "Riemann and the Göttingen School of Physiology", *Fusion*, 6:3 (1989), pp. 24-38. I have also relied on E. Wever & M. Lawrence, *Physiological Acoustics* (Princeton: Princeton University Press, 1954), pp. 145-172, cited in Gallager, *op. cit.*; and Hudspeth, A.: "How the ear's works work", *Nature*, 341, 1989, pp. 397-404. Although Gallager's view of the Riemann-Helmholtz controversy is itself controversial, I agree with his central thesis, that Helmholtz' reductionist approach blinded him to the methodological issue which Riemann was addressing.

8. E. Wever & M. Lawrence, *op. cit.*

9. From Henle's Forward in B. Riemann, *op. cit.*, 1953. Cited in Gallager, *op. cit.*

10. B. Riemann, *op. cit.*, 1953, p. 354; Gallager, *op. cit.*

11. Weber and Fechner are perhaps best known for their formulation of the so-called Weber-Fechner law of psychophysics. They later demonstrated that the perceived loudness of a tone is proportional to the logarithm of its intensity.

12. B. Riemann, *op. cit.*, 1953, p. 343. Cited in Gallager, *op. cit.*

13. E. Wever & M. Lawrence, *op. cit.* Cited in Gallager, *op. cit.*

14. B. Riemann, *op. cit.*, 1953, p. 343. Cited in Gallager, *op. cit.*

15. A. Hudspeth, *op. cit.*, 1989.

16. In Gallager, *op. cit.*, pp 31-38.

17. Riemann, *op. cit.*, 1953, p. 338f.

18. G. W. Leibniz, *Philosophical Papers and Letters*, ed. L. Loemkar, (Chicago, 1956), p 286. From a letter to Herman Corning. Leibniz' differential and integral calculus were well established on the continent in the 19th century, but little of his work was published at that time. Much of his work on methodology is to be found in letters (of which he wrote upwards of 15,000 during his lifetime) and private discourses. Riemann's references to Herbart in both "The Mechanism of the Ear" and in "On the Hypotheses ...", suggest that the former Göttingen professor was a main source of Riemann's knowledge on Leibniz.

    In *The world as I see it*, Einstein – who is steeped in Continental science – makes a similar distinction between *constructive theories* and *principle theories*. *Constructive theories* are "synthetic" in that they:

    > "… attempt to build up a picture of the more complex phenomena out of the materials of a relatively simple formal scheme from which they start out."

    Whereas *principle theories* are "analytic" in that are based on

    > "... general characteristics of natural processes, principles that give rise to mathematically formulated criteria which the separate processes or the theoretical representations of them have to satisfy."

    Einstein, A. (1935), *The world as I see it*, trans. by Alan Harris, London: J. Lane, p. 167)

19. "The Leibniz-Clarke Correspondence" from 1714-1715, in Leibniz, *op. cit.*, 1956, pp. 1085-1169. See especially Leibniz third letter. Evidently, Newton did not wish to openly take part in this debate, and used his friend, Rev. Thomas Clarke, as a go-between.

20. Plato, *The Collected Dialogues*, ed. E. Hamilton E. & H. Cairns, H. (Bollingen Series LXXI, Princeton) 1963, Book VI, §§ 510b-c, 511b-c. This concerns Plato's discussion of the analysis of basic (i.e. ontological) assumptions.