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Ars Morphologica:

Conceptual Modelling, Combinatorial Heuristics and *Ars Inveniendi*

An Epistemological History

Tom Ritchey

Chapters 1 & 2 can be downloaded at:
https://www.swemorph.com/amg/pdf/ars-morph-1-draft-ch-1-&-2.pdf

Chapter 4: Ramon Llull and the Combinatorial Art

> "The understanding longs and strives for a universal science of all sciences, with universal principles in which the principles of the other, more special sciences would be implicit and contained as is the particular in the universal" Ramon Lull – *Ars magna et ultima*.[1]

Ramon Llull (1232 - 1316) is undoubtedly one of the most enigmatic figures in the history of science and philosophy. Both visionary and quixotic, he succeeded in acquiring two nicknames: one the celebrated *doctor illuminates*; the other *vir phantasticus* (crazy man). He was born and brought up under extremely "blended" circumstances at the geographical intersection of two major (hostile) cultures and religions: Christian and Muslim (with the added mixture of Jewish). In addition to this, he lived during a historical-philosophical watershed in which "…no phase of Scholasticism ... remains more confused than the events in the decades immediately each side of the year 1300."[2]

Llull was born in Ciutat de Mallorques (now the city of Palma), on the Mediterranean island of Mallorca, only a few years after the Catalonian kingdom of Argon, under King James I, (re-) conquered it from Muslim rule. During the 1300's it was called "the most nearly Muslim of Christian cities"[3] and it is estimated that at least one third of the population was made up of (more or less enslaved) Muslims. There was also a substantial Jewish population. Note that the Muslim culture of Mallorca at the time of the Aragonese conquest, although hardly on a level with the learned Islamic centres of the Maghreb and Middle East, was nonetheless, in many ways, more sophisticated and learned than the European Christian culture that had conquered it – a fact that engendered both envy and hatred on the side of the Christians.

Llull had an aristocratic upbringing, becoming tutor to King James I's son and later steward of the royal household. However, around the age of 30, after some years of aristocratic overindulgence, he had an epiphany: he would dedicate his life to Christianity and, in particular, to the conversion of the Muslims (and Jews) to the Christian religion. However, he was decidedly different from most of the other "converters" of the time: he was totally against the use of coercion in this endeavour and declared that any effort to convert Muslims and Jews to Christianity would remain fruitless if Christian missionaries could not give them a good (i.e. *rational)* reason to do so.

In order to carry out this enterprise, he would need to be able 1) to speak and write Arabic and understand Islamic culture and theology, and 2) to have at his disposal a pedagogical method that was not based on "authoritative Christian texts" (scripture) and purely dogmatic beliefs, but on rational and *theologically neutral* arguments. He needed a program for the art of interreligious dialogue, where "everyone could use the same thinking alphabet and the same speaking alphabet".[4] His aim was:

> "… to find a mechanism to prove and generate truths in such a way that, once everyone agreed on the [initial] assumptions, the objectivity of the procedure would force all to accept the conclusions."[5]

To this end, Llull "acquired" a literate Muslim slave and spent nine years learning the Arabic language and culture in preparation for his mission. During this period he also studied Arabic science and literature, including the Neoplatonic texts of Al-Farabi, Ibn Sina (Avicenna) and Al-Tusi.[6] He also studied ancient Greek philosophy including (importantly) the Neoplatonic texts of John Scotus Erigena and Pseudo-Dionysus, the latter a Christian interpreter of Proclus, the most influential of the ancient Neoplatonists for early modern science.[7] This "private education" resulted in Llull being forever an academic outsider, but at the same time it left him academically unfettered to devise his own methods and tools.

It has been estimated that Llull wrote some 280 books – many of them extensive – in Latin, Catalan and Arabic. The works were quite varied, including romantic novels, poetry and a dictated autobiography. However, his ultimate goal always remained the same.

> "He wanted to write a book which would make Christian doctrines intelligible to Moslems and Jews. He called his book the *Ars inveniendi veritatem*, the Art of Finding the Truth ... and worked unflaggingly on the composition of this Art for more than thirty years."[8]

4.1 Llull and *ars inveniendi*

In order to be able to argue rationally about the articles of Christian faith, Llull could not start from premises based on sacred script or Christian theological dogma, as these were exactly the point a contention. He needed theologically neutral premises – concepts which could be agreed upon by all three faiths – from which he then could carry out rational demonstrations. But how does one go about developing such premises?

Scholastic "science" of the 13th and 14th centuries (for Llull represented primarily by the academic world of Paris) was dominated by rhetorical "logic" and the method "deductive proof" (*ars demonstrandi*) based on Aristotelian syllogism. However, in order to *demonstrate* a "truth" you first had to have – or *find* – true premises. In this context, Llull had two misgivings about what he considered a stultified Scholastic logic:

> "... on the one hand, the insufficiency of the demonstration by syllogisms – the basis of all Scholastic science – to reveal new truths, because it only explicitly states the relationship between known facts and evident principles, thus relegating dialectics (or *ars inveniendi*), which consists of *finding arguments* and counterarguments based on some *loci* or "places", to the domain of *opinion*. On the other hand, this criticism addresses the fact that demonstration by syllogisms only works through second intentions; that is, it describes relation-

> ships within logical propositions [*in intellectu*] and not according to objects in reality [*in re*], which are conceptualized as first intentions."[9]

Llull needed to re-axiomatize Western monotheist theology by finding basic premises from which valid logical demonstrations could be inferred, thus *combining* the reciprocal methods of *ars inveniendi* and *ars demonstrandi* [see §2.2 *supra*].

> "If now we try to analyze the techniques Llull is using here, we will see first of all that he is not ... working according to the Euclidean model, where a group of *pre-established* principles are used to prove successive new principles (theorems) [the geometric or axiomatic-synthetic method]. Instead he presents the question as a *hypothesis*, and draws out the implications of assuming this hypothesis to be true or false. The positive one merely shows that the argument is valid, and that therefore the premise leads to no inconsistency, whereas the negative form uses the classic *reductio ad absurdum* to show that if the negation of a premise leads to an impossibility, this proves the premise. Rather than using principles to work *towards the thing to be proved* [synthetically], he is, as it were, *working backwards*, starting with the hypothesis to test it against the principles [analytically]."[10]

This is Plato's hypothetical-analytic (and Peirce's abductive or retroductive) method, i.e. the "analytic" phase of *sequential* analysis and synthesis [§2.3 *supra*]. Then Llull does something new: he combines this with *compositional* analysis and synthesis such that

> "... each complex concept that we conceive is analyzable into all its component parts, down to the most simple ones. Simple, i.e. un-analyzable, concepts are the 'primitives' from which all compound concepts originate. Once one arrives at the primitives, one may choose an alphabet and attribute a sign to each primitive: thus, by combining signs, one will obtain all the complex concepts in a purely mechanical way. *Analysis, synthesis*, and the discovery of an appropriate alphabet were the main stages of this program aiming to renew logic as both an art of judgement (the *ars iudicandi* [*demonstrandi*]) and an art of discovery (*ars inveniendi*).[11]

> "... starting on the basis of categories of universal application and operating with a system of symbolic notations and combinatorial diagrams, Lull establishes the principles of a *synthetic and inventive procedure* which, unlike the demonstrative logics of Aristotle, would not be limited to ... truths already known but would make possible ways for the discovery of new truths."[12]

As the eminent Llull scholar Anthony Bonner has observed, "... Llull's invention of an ***ars combinatoria*** *as the only possible way of dealing with interrelationships of Platonic forms* was to have a considerable impact in the Renaissance ... and would have a decisive influence on Leibniz".[13]

4.2 Islamic influence on Llull's combinatorial art

The classical Greek mathematicians seem to have been more enamoured of *geometry* (i.e. the mathematics of magnitude and the continuous) than of discrete mathematics (i.e. of multitude and combinatorics).[14] Although there has been some debate on the matter of just how much the Greeks concerned themselves with combinatorial studies[15], the fact remains that little work in this area *per se* was passed down directly from ancient Greek and Latin sources to medieval and Renaissance Europe. Indeed, one of the latest anthologies on the subject – "Combinatorics: Ancient and Modern"[16] – contains introductory sections on *Ancient Combinatorics* which take up Indian, Chinese, Islamic and Jewish contributions, with the conspicuous absence of any significant Greek or Latin contributions.[17]

Indeed, the popularity of combinatoric methods emerging during the Renaissance was based *primarily* on Llull's work.[18] So from where did Llull get it?

> "In view of the almost total absence of relevant material in classical Greek and Latin literature, we must look elsewhere for the source of basic combinatorial lore. The Chinese may have a claim, but the available evidence seems to indicate that the main stimulus came from another Eastern people – the Hindus."[19] … the Hindus were accustomed to the idea that complex objects and concepts arise from combinations of more basic things, and so the mathematical questions occurred naturally in their scheme of discovery."[20]

The earliest example of combinatorial mathematics is usually ascribed to the ancient Indian mathematician Pingala in the *Chandaḥśāstra* (or *Pingala Sutrasc*) from c. 200 BCE. What drove this development was "a phonologically structured syllabary such as that of Sanskrit for a mathematical study of prosody, the science of *chandas*".[21] That is, the study of permutations and combinatorial methods of enumerating poetic and musical metre in a syllabic script (see below). Indeed, the letters of a phonetic script – whether syllabic or alphabetic – quite naturally lend themselves to combinatoric analysis (as with Plato and his epigones somewhat earlier).

During its 7th century expansion, Islamic culture came into contact with Hindu arts and science, which eventually allowed Islamic mathematicians to combine Hindu and classical Greek mathematics. Thus, in eastern Arabic territories we find the expression for "arithmetic" being called *ḥisāb al-hindī* (Hindu reckoning).[22]

> "With regard to simple combinatorial principles, the Arabs seem to have acquired the techniques used by the Hindus and applied them in similar ways. An interesting example occurs in the work of al-Halil Ibn-Ahmad (718-791 AD) … [who] considered the *possible arrangements of letters in the formation of syllables*, and his calculations show that he understood the basic formulae for finding numbers of permutations and combinations."[23]

During the Islamic Golden Age (c. 750 - 1300), Islamic philosophers and mathematicians further developed all of the constituents that would show up in Llull's Great Art. This includes:

- the beginnings of a *mathematical theory of combinations and permutations*;
- the use of an *abbreviative letter symbolism*; and
- the appreciation of the *reciprocal nature* of discovery (*ars inveniendi*) and proof (*ars demonstrandi*), including their connections with *analysis* and *synthesis*

This last development was especially important:

> "... Ibrahim ibn Sinan developed a pragmatic logic in which he coordinated an *ars inveniendi* with an *ars demonstrandi.* He was followed by al-Sijzi, who wrote an *ars inveniendi* based on Thabit ibn Qurra and his grandson. Then came Ibn al-Haytham, who developed an analytical art, and, later on, al Samaw'al, who examined *analysis* and *synthesis* in algebra. *The world of mathematics would have to wait until late in the 17th century, with Leibniz, for any contribution of comparable importance*."[24]

Finally, there is the *Zairja*. As early as the 10th century the Islamic philologist and grammarian Ibn Durayd (837-933), wrote of a method of combining letters in different ways by arranging them in concentric rings and then turning the rings to produce different combinations.[25] Llull knew of this "divinatory device" called a *Zairja* (*zā'irjah*) as attributed to a Maghribi Sufi at the end of the twelfth century.[26] It was constructed from paper and could be manipulated to process symbols in order to produce many different word combinations from a relatively small initial set of letters. Ibn Khaldūn described this as "a branch of the science of *letter magic* … the technique of finding out answers from questions by means of connections existing between the letters of the expressions used in the question."[27] Such "question-answering disks" were known to be on sale in marketplaces in Algeria in the 1260's and 70's.[28]

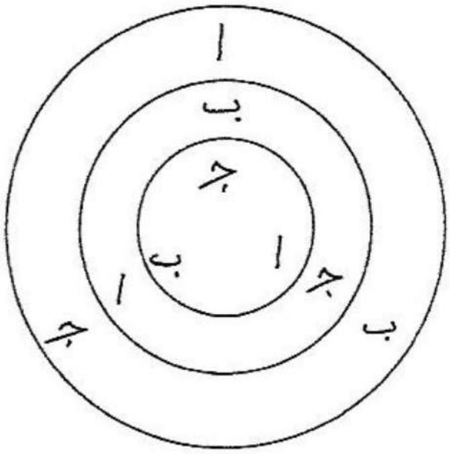

Figure 4.1: Rotating rings of Ibn Durayd (from Djebbar, 2013)

Such "letter magic" is especially suited to Arabic *abjad* (syllabic or consonantory) scripts, which omit some or all vowels. This means that the output of such a "combinatoric calculator" can produce many more *meaningful* – even if "fantastical" – interpretations of symbol combinations and permutations than one could produce in a more constrained "vowel alphabet".[29] A vowelless syllabic script has a greater potential for *combinatoric poiesis*.

In 16th and 17th century Europe, Llull's movable *Zairja* had become "an intellectual obsession in European culture".[30] They were known as *Volvelle* – literally "revolving vellum" – and were incorporated in so-called "movable books" as mechanical learning devices.

This background in Islamic influence is not meant to diminish Llull's achievements. He adopted and adapted these themes from Islamic mathematics and logic first into theology, and then attempted – or at least proposed – their generalisation into a heuristic modelling procedure for *science as a whole*. Furthermore, as concerns the *Zairja*, Llull was not interested in combinatoric *word poiesis*; he was interested in combinatoric *proposition poiesis*. He wanted to generate new propositions that could then be ordered and employed as premises for deductive inference.[31] He wanted to produce an *inveniendi-demonstrandi* "logic-machine". Indeed, in face of the historical evidence of his influence on Renaissance and Enlightenment science, Francis Yates went so far as to say: "The European search for method . . . began with Ramon Lull."[32]

4.3 The Great Art

Llull's point of departure for his Great Art was not Christian scripture, but a hypothesis about – and an analysis of – the most basic theological principles which are common to all three Western monotheistic religions. At the head of these first principles were the "primary attributes" or "dignities" of God which are found in the Christian works of Pseudo-Dionysius[33], in the Islamic *Hadras* and the Judaic *Sefirot*, all of which were heavily influenced by Neoplatonic tradition.[34]

> "The religious principles upon which Lull based his Art, which were held by all three religious traditions, was the importance which Christian, Moslem, and Jew attached to the Divine Names or Attributes. The Attributes, or, as Lull prefers to call them, the *Dignities of God* on which the Art is based are Bonitos (Goodness), Magnitudo (Greatness), Eternitas (Eternity), Potestas (Power), Sapientia (Wisdom), Voluntas (Will), Virtus (Virtue or strength), Veritas (Truth), Gloria (Glory). Religious Moslems, Jews and Christians would all agree that God is good, great, eternal, powerful, wise, and so on. These Divine Dignities or Names, combined with elemental theory, gave Lull what he believed to be a universal religious and scientific basis for an Art so infallible that it could work on all levels of creation."[35]

Symbols	First figure (Figure A)	Second figure (Figure T)	Questions and rules	Subjects	Virtues	Vices
B	goodness	difference	Whether	God	justice	avarice
C	greatness	concordance	what?	angel	prudence	gluttony
D	eternity or duration	contrariety	of what?	heaven	fortitude	lust
E	power	beginning	why?	man	temperance	pride
F	wisdom	middle	how much?	imaginative	faith	accidie
G	will	end	of what kind?	sensitive	hope	envy
H	virtue	majority	when?	vegetative	charity	ire
I	truth	equality	where?	elementative	patience	lying
K	glory	minority	how? and with what	artifice	pity	inconstancy

Figure 4.2: The "Alphabet" – Llull's master morphospace

Llull then establishes a field of six classes of concepts which he called "Principles", each containing nine categories (Figure 4.2). This serves as his "master morphospace" for which he can sort out different sets of "Principles" to interrelate combinatorially. He gives each of the nine categories of each "Principle" a common label (alphabetic sign, at the far left) which he can then use to symbolize them into combinatory configurations. This is what he calls his "Alphabet". The "Divine attributes" (Figure A: second column) are treated as subjects of predication, while the other five columns are treated as predicates. (Although the *contents* of this "Alphabet" may strike us today as both arbitrary and trivial – which is exactly what Leibniz thought about them some 350 years later – in the theological context of Llull's time they were considered essential.)

Binary combinatorics

Llull applied combinatorics to his Alphabet in several ways. Firstly, in the simplest case, the First Figure (Figure A) can be combined with itself if one treats it variously as subject and predicate, as shown in Figure 4.4 (below). Llull writes: "This Figure is circular to show that any subject can become a predicate, and vice versa, as when one says, 'goodness is great', 'greatness is good', and so on."[36] (Combinatorial repetition, for instance "BB" or "Goodness is good", is not allowed.)

Figure 4.3: Llull's first "Figure", denoted by A

Symbols	Figure A Subject	Figure A Predicate
B	goodness	good
C	greatness	great
D	eternity	eternal
E	power	powerful
F	wisdom	wise
G	will	wilful
H	virtue	virtuous
I	truth	truthful
K	glory	glorious

Figure 4.4a: Llull's combinatory matrix

BC	CD	DE	EF	FG	GH	HI	IK
BD	CE	DF	EG	FH	GI	HK	
BE	CF	DG	EH	FI	GK		
BF	CG	DH	EI	FK			
BG	CH	DI	EK				
BH	CI	DK					
BI	CK						
BK							

Figure 4.4b: Llull's 36 cell combinatory matrix for Figure A as subject-predicate

The possible combinations of nine elements taken two at a time, when the inversion of order is allowed, but when repetitions are not allowed, is (36x2=) 72 propositions.

Trinary combinations

The so-called Fourth Figure (Fig. 4.5) consists of three "principles" in combination. Given the 9 variables (B through K), there are 9^3 possible ternary combinations if one allows for repetitions. However, *without* repetitions we get

$$\frac{9!}{3!(9-3)!} = 84$$

ternary combinations (Figure 4.6).

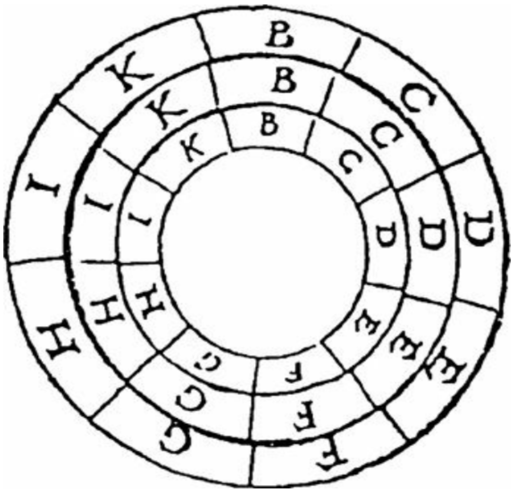

Fig 4.5: Quarta Figura

1 BCD	2 BCE	3 BCF	4 BCG	5 BCH	6 BCI	7 BCK	8 BDE	9 BDF	10 BDG	11 BDH	12 BDI
13 BDK	14 BEF	15 BEG	16 BEH	17 BEI	18 BEK	19 BFG	20 BFH	21 BFI	22 BFK	23 BGH	24 BGI
25 BGK	26 BHI	27 BHK	28 BIK	29 CDE	30 CDF	31 CDG	32 CDH	33 CDI	34 CDK	35 CEF	36 CEG
37 CEH	38 CEI	39 CEK	40 CFG	41 CFH	42 CFI	43 CFK	44 CGH	45 CGI	46 CGK	47 CHI	48 CHK
49 CIK	50 DEF	51 DEG	52 DEH	53 DEI	54 DEK	55 DFG	56 DFH	57 DFI	58 DFK	59 DGH	60 DGI
61 DGK	62 DHI	63 DHK	64 DIK	65 EFG	66 EFH	67 EFI	68 EFK	69 EGH	70 EGI	71 EGK	72 EHI
73 EHK	74 EIK	75 FGH	76 FGI	77 FGK	78 FHI	79 FHK	80 FIK	81 GHI	82 GHK	83 GIK	84 HIK

Figure 4.6: Table of ternary combinations

One way of using the table is to treat a single "principle" internally. So, for cell #1 (BCD) using the Divine Attributes (Figure **A)** alone:

BDC = Goodness is as Great as Eternity

One can also combine across variables, using e.g. Figure **A**, Figure **T** and **Virtues** in turn:

BDC = Goodness is in Concordance with Fortitude

Note also that using three rings (i.e. variables) lends itself to the formulation syllogistic inference, i.e. a major and minor premise, and a conclusion. Of special interest is the possibility of proving a given conclusion on the basis of a given major premise by finding an appropriate "middle term": one chooses a relationship between the outer and inner rings, and then rotates the middle ring to see if any of the combinations form a valid inference.[37] This was one of Llull's primary goals, to be able to "prove" correct theological doctrine deductively by systematically going through all the combinatoric possibilities.[38]

The number of three-term propositions that can be formulated using all of the six principles, with the "Divine attributes" (Figure A) treated as subjects of predication, and the other five as possible predicates, is 7290. Of course, many of these combinations might make little sense, as Llull well understood. It is the responsibility of the user to weed out meaningless propositions.

Since we are mainly interested in the epistemological principles involved, we have presented only a few of the most basic operations in Llull's work, which he developed into a hugely complex system. To get a taste of just how complex the "Art" became, the reader is referred to Antony Bonner's excellent "The Art and Logic of Ramon Llull - A User's Guide".

Llull also experimented with other ways to represent the interrelationships between Platonic forms, as shown in Figure 4.7. This "Euler-diagram" – in one of its earliest known uses – is found in a late copy (from 1617) of Llull's "Opera ea quae ad adinventam", a work concerning inventive methods.[39] It is meant to convey the idea that that nothing exists (Esse) which does not possess unity (Unum), truth (Verum) and goodness (Bonum). (Note that this is not a "full" 4-species typology, since this cannot be represented "completely" using circles. Cf. figures 2.7a & 2.8a, § 2 *supra*.)

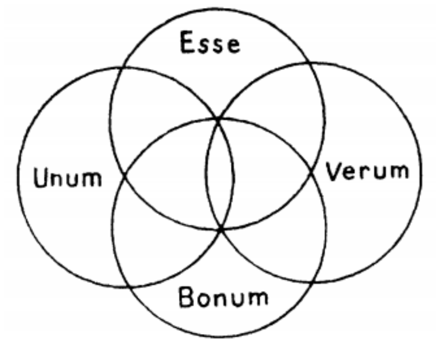

Figure 4.7: Llull's use of "Euler diagram" for typological structure

Although Llull's main application of the combinatoric methods was concerned with theological and philosophical questions, he did apply it to more mundane matters, two examples being medical diagnostics and voting theory. In *Liber Principiorum Medidinae* (1275) he proposed using the combinatory art as a diagnostic system to relate symptoms to treatments[40] – which is what we do today with computerised "expert systems". And in *De Arte Eleccionis* (1299) he developed the earliest known version of a "rank-order-count" voting system[41], which today is generalised in the form of Multi-Criteria Decision Support systems such as the Analytic Hierarchy Process (AHP).

From the perspective of the history of philosophy of science, it is also interesting to note the different attitudes toward Llull's work by different theological and philosophical traditions of his time. As Frances Yates has noted, Llull's analytic-synthetic "combinatorial art" represented a novel *competing method* to the classical Art of Memory (*Ars Memoriae*), which was based on the "method of places" (mnemonic *loci*) using mental imagery.[42] In contrast, Llull's art was an early version of a mechanically mediated system of "extended cognition".

> "It is a fact of some historical importance that the two great mediaeval methods, the classical *art of memory* in its mediaeval transformation and the art of Ramon Lull, were both rather particularly associated with the mendicant orders, the one with the Dominicans the other with the Franciscans."[43]

Now it is documented that Llull approached the powerful Dominican order in hopes of getting its support for his method, but without success. At this point in time, the Dominicans had recently refurbished their own methods. In the early 1260's, Albertus Magnus (c. 1200-1280) and his pupil Thomas Aquinas (1225-1274) established their *ratio studiorum*, an educational program aimed at harmonizing Aristotelian teaching with the doctrines of the church, thus initiating the tradition of *Dominican scholastic philosophy*.[44] This further established Aristotelian syllogism and *ars demonstrandi* as the approved instrument of Scholastic reasoning.

> "On 19 March 1255 the most influential faculty of arts of Paris stipulated the study of all the writings of Aristotle that were known at the time... One could say that this acknowledgment of Aristotle's philosophy constituted the birth of the faculty of arts, namely of its independence vis-à-vis theology. Soon it became the norm everywhere, and in many places remained so until the end of the Middle Ages, that Aristotle was the core, and often the exclusive content, of the study of philosophy." It was seen as "... a coherent set of dogmas and therefore as something that is finished and completed; thus, philosophy is only a small step away from dogmatism, the mortal enemy of living thought."[45]

Llull's heuristic program of *externalising* "place imagery" – thus disburdening internal memory – and analysing Christian doctrine in order to generate premises which could result in new possible deductive consequences, found little interest among the Dominican establishment and many of his works were later banned by the Dominican Inquisitor General. The Franciscans, on the other hand, were more interested in his ideas, as their theological orientation – through Augustine, Anslem and Scotus Erigena – was more attuned to the Neoplatonic tradition. Llull eventually became a Franciscan tertiary, a secular member of the Franciscan order.[46]

To sum up: Llull's work has been considered by some as being foundational for Western *combinatorial mathematics* and even for the development of symbolic systems. However, this both overestimates and underestimates him. Firstly, Llull's art was not a true *operative symbolic system*: his use of letter-signs was totally abbreviative [see 2.5 *supra*]. Still worse, each individual "sign" denoted several different attributes at the same time, thus totally defeating the purpose of an operative symbolic system. And although he did draw attention to the *mathematical* aspect (and got plenty of it wrong), he was far more interested in his Art as a method of "inventive logic".[47]

> "One of the principal innovations of the Art was the desire to fuse two fields that had until then been treated separately: the *ars inveniendi* of dialectic and the *ars judicandi* of logic ... and to create a system that could deal with both."[48] ... "Moreover, the combinatory mechanisms with their accompanying graphic devices made possible what was perhaps the most innovative of Llull's accomplishments: to create an Art that was *generative*, which upon a base of a strictly limited number of concepts could build a whole constellation of demonstrations and explanations. And it was this *generative nature* of the Art which held such a powerful fascination for later thinkers."[49]

Llull's basic idea of systematically inter-relating Platonic concepts in order to generate all possible ("constructive") combinations – thereby identifying new, previously unknown "blended" concepts – was something new.[50] Firstly, it showed that the *synthesis phase* of compositional analysis and synthesis had its own function as an *ars inveniendi*. And the same system that can generate such combinations can also be turned around to test for their validity (*demonstrandi*).[51] It was these central methodological features of Llull's Art that would attract Gottfried Leibniz, who would develop "universal analysis and synthesis" into a model-based art of discovery.[52]

Outline of research:

Ars Morphologica: A History of Model-Based Reasoning and the Art of Discovery from Plato to Zwicky

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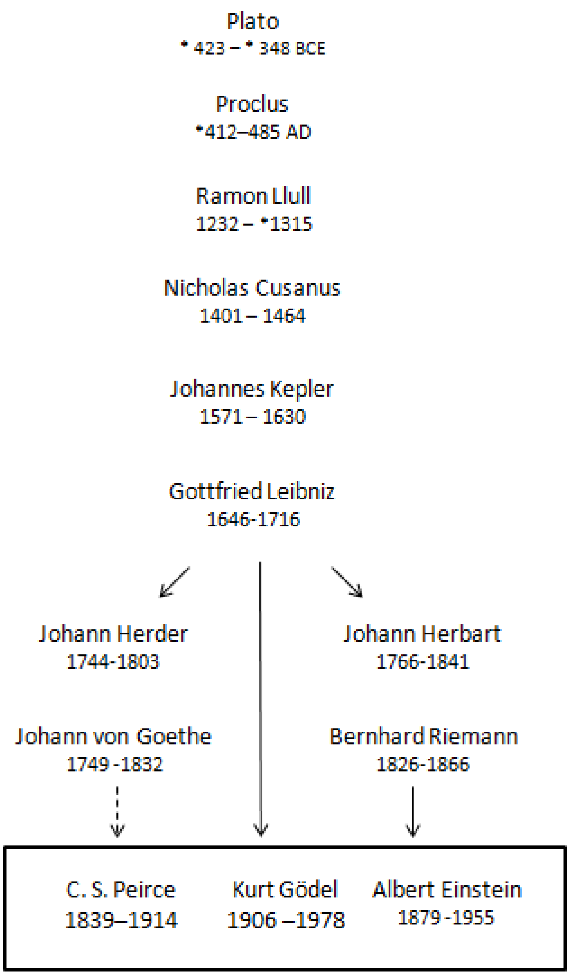

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Notes

[1] Cited in MacLennan (2021), p. 83.

[2] Left (1976), p. 32, cited in Johnston (1987). Cf. Yates (1982), p. 9: "[T]he great thirteenth century which saw the development of scholasticism out of the re-discovery of Ariastotle."

[3] Cited in Simon (1998), p. 25.

[4] Seghedin (2007), p1.

[5] Sale (2011), p. 26f. The great Llull scholar Anthony Bonner puts it a bit more cynically: "[The] trick was to ensnare his interlocutors with unobjectionable foundations, whose consequences they would then be unable to refute." Bonner (2007) p. 284.

[6] See Rescher (1962) concerning Al-Farabi's promotion of non-Aristotelian inference and the epistemological distinction between generating ideas and demonstrating proofs.

[7] Cf. Yates (1966), pp. 177-179.

[8] Lohr (1992), p. 24

[9] Priani (2017) (my brackets). Cf. Yates (1954, p. 156) for more on Llull's distinction between first and second "intensions".

[10] Bonner (2007), p. 81. (My emphasis and brackets).

[11] Mugnai (2010), p. 302.

[12] Pombo (2010), p. 1. Note that Llull's signs are purely abbreviative and do not constitute a true operative symbolic system. [See § 2.5.2 *supra*]

[13] Bonner (1997), p. 14 (My emphasis).

[14] Biggs (1979), p. 113: "In any discussion of the antiquity of combinatorial calculations, the contribution of the ancient Greeks must be assessed. The assessment is remarkably negative. There are very few relevant remarks in the whole of extant Greek literature …".

[15] Recent research has shown that there was indeed ancient Greek works on combinatorics, including those of Chrysippus (279-206 BCE), Hipparchus (190-c. 120 BCE) and Archimedes (c. 287 – c. 212 BCE), but they were not preserved and transmitted. See e.g. Acerbi (2003); Bobzien (2011). Also, Nicomachus (c. 60- c. 120 AD) is regarded as a number theoretical specialist, but this is primarily from the perspective of Pythaean Theosophical Numerology, not technical combinatorics. Cf. Kahn (2001), pp. 110-118. Cobeli & Zaharescu (2013) suggest that Apollonius of Perga (3rd Century BCE) knew of the Arithmetic Triangle, which is a sure mark of combinatorics knowledge.

[16] Wilson & Watkins (Eds.) (2013).

[17] However, there is one area of interest which seems to be universal, and may represent the independent geneses of combinatorial curiosity across many cultures: Language. Both spoken and written language is inherently combinatoric. And this is one area that Plato and the Classical Greek philosophers did treat. In the Sophist, Plato remarks about how different letters of the alphabet can and cannot combine. Xenocrates (396 - 314 BCE), Plato's close friend and head of the Platonic Academy from c. 339 to 314 BC, made an attempt to calculate the total number of syllables which could be constructed from the letters of the (Greek) alphabet. The result which Xenocrates obtained was, according to Plutarch, 1,002,000,000,000. This possibly "represents the first [European] attempt on record to solve a difficult problem in permutations and combinations." See e.g. Heath, 1981, p. 319. (Cited in Stanley, 2011 p. 105). Another associated area of general interest for combinatorics concern musical and poetic *meter*.

[18] For a historical background, see Knoblock (2013). Cifoletti (1996) has documented how Renaissance scholars systematically searched for classical (Greek and Latin) sources in an attempt to diminish the importance of Islamic influence on European mathematics.

[19] Biggs (1979), p. 114.

[20] Ibid. p.133f.

[21] Divakaran (2016) p. 64.

[22] Oaks (2012), cited in Heeffer (2012). The Persian mathematician Kushyar ibn Labban wrote *Principles of Hindu Reckoning* (*Kitab fi usul hisab al-hind*) in the 11th-century

[23] Biggs (1979) , p 117f (Emphasis added)

[24] Rashed (2009), p. 12 (Emphasis added)

[25] Djebar, 2013, p. 85. Such devices were reportedly “on sale” at bazaars.

[26] Link 2010, p. 217.

[27] Gray (2018), p. 297. Ref: Ibn Khaldūn (1958). My emphasis.

[28] Sales (2011) p.32.

[29] Link (2010), p. 217.

[30] Rossi (2000) , p. 29.

[31] Link (2010),p. 216. “…the disk construction served theoretical functions as an encyclopaedia of religious thought, a tool to inspire meditation about its main topics, and a way to *generate new propositions*.”

[32] Yates (1982), p. 7.

[33] “[The] Names of God belong very strongly into the Christian tradition; many of them are mentioned by Augustine, and in the *De divinibus nominibus* of Pseudo Dionysius they are listed at length. The names used by Scotus Erigena (c.815–77) and by Ramon Lull are nearly all to be found in the book *On the Divine Names* of Pseudo Dionysius” [i.e. the anonymous author who incorporated the Neoplatonic doctrines of Proclus into Christianity.] (Yates (1966), P.177)

[34] Cf. MacLeenan (2007).

[35] Yates (1979), p. 13.

[36] Cited in Bonner (2007), p. 125f.

[37] See e.g. Bonet (2011).

[38] See Welch (1990) for a more detailed exposition.

[39] Baron (1969), p. 115.

[40] Wilson & Watkins (2013), p. 16.

[41] McLean & London (1992).

[42] See “Lullism as an Art of Memory”, in Yates (1966).

[43] Ibid. p. 175.

[44] “On 19 March 1255 the most influential faculty of arts of Paris stipulated the study of all the writings of Aristotle that were known at the time... One could say that this acknowledgment of Aristotle’s philosophy constituted the birth of the faculty of arts, namely of its independence vis-à-vis theology. Soon it became the norm everywhere, and in many places remained so until the end of the Middle Ages, that Aristotle was the core, and often the exclusive content, of the study of philosophy.” Furthermore, this philosophy was seen “as a coherent set of dogmas and therefore as something that is finished and completed; thus, philosophy is only a small step away from dogmatism, the mortal enemy of living thought.” Höffe (2003), pp. 195f.

[45] Höffe (2003), pp. 195f.

[46] "Lull was not a scholastic, he was a Platonist, and in his attempt to base memory on Divine Names, which verge on Platonic Ideas in his conception of them, he is closer to the Renaissance than to the Middle Ages." Yates (1966), p. 176.

[47] Knobloch (2013): "Unlike his followers in the 17th century, Llull was not primarily interested in the mathematical problem of enumerating the possible sign combinations. His combinatorial art was designed as an inventive logic that could judge every posed problem." (p. 125).

[48] Bonner (2007), p 296. "One could even say that it is an essential feature of the Art, one that makes the Art itself an exemplary structure. It thus can generate not only answers or demonstrations, but even the questions themselves! It is therefore "inventive" in a much broader sense than the normal medieval meaning derived from the Topics, of "finding" strategies for dialectical arguments; Llull's suggestions for its use as an exemplary, self-generative structure, makes it possibilities almost limitless. The ability to apply the Art beyond its own borders make it what Umberto Eco has called an opera aperta, one in which the user is invited to continue the work begun on the pages he has before him. (Ibid. p. 195f.)

[49] Ibid, p. 290. Cf. Höffe (2003), p. 25: "In opposition to a Scholasticism gone sterile, and content with an art of proving, an *ars demonstrandi*, the alternative, an *ars inveniendi*, accompanied the great scientific awakening of the early modern era. From Bacon to Vico, Wolff and Lambert, by way of Descartes and Leibniz, philosophers in general were looking for a new 'tool'."

[50] "Llull's invention of an ars combinatoria as the only possible way dealing with interrelationships of Platonic forms, was to have considerable impact in the Renaissance ... and a decisive influence on Leibniz." Bonner (1997), p. 18.

[51] "Perhaps the most striking of Llull's anticipations was the idea of having a *finite* set of *rules* as well as a finite set of truths —"basic concepts", *axioms* or whatever you call it—, so that you can then generate from them a (presumably infinite) set of *derived* truths. Nowadays we would describe the idea more simply, and say that Llull had just come across the idea of a *generative* system. In linguistics such a finitistic device is called a *grammar* ... and the generated strings are the *language*. In Computer Science the device is called a *machine* and what is being generated is the set of output configurations ... As is well known today, the same mechanism can run backwards: the same grammar that is capable of *generating* a language is also capable of *accepting* or *recognizing* its strings as belonging to it. Or the same machine which computes the batch of acceptable results is also capable of recognizing a correct calculation." (Sales, 2011, p. 29.)

[52] "Leibniz's thoughtful 1666 'Dissertatio de arte combinatoria' is not only good and interesting reading for today's logicians and mathematicians. It is the best criticism and homage that Llull has ever received: by recognizing his merits and adapting his ideas to the modern needs of Science, Leibniz did all to include Llull in our scientific heritage, and did us a favor in the process." (Sales, 2011, p. 33.)