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In the Beginning was Chiasmus

On the epistemology of non-quantified modelling

By
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Synopsis: *Chiastic order* is an ancient expression for cross-classification. As such, it is the primordial form of non-quantified modelling and combinatory heuristics. This book presents an epistemological history of non-quantified modelling: its prehistory in the form of rhetorical and epistemic *chiasmus*; its early (pre-symbolic) use by Plato as a cross-order (*paradigmatic*) modelling method; its prototype development by Ramon Llull as a heuristic *ars combinatoria*; its "modern" use by Gottfried Leibniz based on a calculus of concepts; and Fritz Zwicky's 20th-century promotion of the "morphological approach". Examples of contemporary computer-aided morphological modelling are presented in the areas of operational research, policy analysis, scenario development and design theory.

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1. Introduction - The Science of Cross-Classification

1.1 20th Century background

In 1906, Alfred North Whitehead (1861–1947) published an article – "The Axioms of Projective Geometry" (**APG**) – in which he made a curious association between geometry, classification theory and combinatoric (heuristic) modelling. Note that by the late 19th century, projective geometry had become a significantly more abstract mathematical discipline than its original conception as the study of perspective and geometric projection *per se*. It had developed into a highly abstract, non-metric form of geometry that deals with inter-dimensional relationships between geometric concepts and their symbolic representations.[1] Thus in **APG**, Whitehead wanted to develop a symbolic system for a geometry without magnitudes and independent of figures.[2] He called this "the science of *cross-classification*".

> "It is well-known that Geometry can be developed without any reference to measurement – and thus without any reference to distance, and without any reference to numerical coordinates for the indication of points. Geometry, developed in this fashion has been termed 'Non-metrical Projective Geometry'. … I have termed it, *'the science of cross-classification'*." (Whitehead, 1933, p. 137f)[3]

Then he says something very interesting:

> "[The] science of cross-classification ... is not a science with a *determinate subject matter*. It is concerned with *any subject matter to which the formal axioms may apply*." (Whitehead, 1911, p. 30)[4]

So what is Whitehead up to? In what way is **APG** "the science of cross-classification" and what does it mean that this "science" does not have any "*determinate subject matter*"?

First of all, Whitehead was involved in the early 20th century European "Leibnizian revival", which will be discussed in more detail in §7. Associated with this were two important developments taking place during the second half of the 19th century which engaged Whitehead in his early carrier as a mathematician (and indeed which evidently had significant ramifications for the whole of Victorian culture[5]). One was a revolution in the concept of *space* and a corresponding generalization of geometry – "projective geometry" being one such generalization.[6] The other concerned the development of *operative symbolic systems* which had been initiated on the "Continent" in the 17th century, and further developed in the works of e.g. Boole, De Morgan, Peacock and Peirce.[7]

Whitehead was working at the intersection of these two areas of mathematics. Just as one can use the rules (axioms) of arithmetic to create symbolic systems for quantitative operations (i.e. the algebra of Viète and Descartes), then if one can formulate a rigorous axiomatic for a *non-metric* geometry, one should be able to use such axioms to create symbolic systems for non-metric relationships, an "algebra of concepts".[8] Like its quantitative cousin, such an algebra should have the potential to become a heuristic tool which is not fixed to any determinate subject matter, but which can "model" anything that is consistent with its axiomatic rules.[9]

In this endeavor, Whitehead is not only plugging himself into an epistemological tradition emerging out of the 17th century "symbolic revolution"[10], but also into a pre-symbolic tradition stretching back to antiquity. We can see this more clearly if we follow his subsequent development from mathematics and logic into the philosophy of science. During the period between 1906 and 1947 (the year of his death), he wrote a number of books and articles in which he further discusses and develops the epistemology of *cross-classification*.[11] His remarks reveal not only something about the concept *per se*, but also about its epistemological history and its association with *conceptual modelling* in general[12]. Indeed, he finds its ancient roots in the Platonic dialogues, and its modern roots in the works of Gottfried Leibniz (1646–1716).

Along the way, he has some interesting things to say about the distinction between two fundamental, perennial classificatory systems: *Linear Hierarchal Classification* (LHC) and *Combinatoric Cross-Classification* (CCC), including the different types of reasoning involved in these two epistemic structures. Remarkably, over the four decades of Whitehead's work in this area, a simultaneous development was taking place in several other scientific disciplines, where this very same distinction was being explored and applied. This included early 20th century developments in bibliographic classification theory[13], in archaeological methodology[14] and in the structuralist approach to narratology and linguistics – e.g. from Vladimir Propp's classification of folktales to Greimas' *Semiotic square* [see §2]. This fundamental distinction between LHC and CCC also turns up – rather inconspicuously – in contemporary ISO terminology standardization under the rubric of "concept relations"[15]

- ***"Classical" (Platonic pre-symbolic) roots***

In "Adventures of Ideas" (1933) Whitehead contrasts the "science of cross-classification" with its contrary, the "science of *mutually exclusive* classification", i.e. the strict classification of genera, species and sub-species, which he attributes to Aristotle.[16] At first, he simply states that this (historically dominating) hierarchal classification approach was based on Plato's "method of divisions" (*diairesis*[17]). However, in "Essays in Science and Philosophy" (1947), he revises this and points out that Platonic *diairesis* also involved another type of classificatory structure – called *cross-division* – based not on hierarchal *sub-ordination*, but on conjunctive *co-ordination* – or the "co-mingling of classes". This is, he states:

> "... a wider conception of the scope of Logic which was obscured by the dominant Aristotelian [linear-hierarchal] theory. The [wider] concept was adumbrated by Plato, when in *Sophist* he points out the importance of a *science of the mingling of forms* [by which] to obtain *more complex forms*."(Whitehead, 1948, p. 237f. Emphasis added)

He cites the following lines from *Sophist* (253d-e):

> "Now since we have agreed that the classes also commingle with one another or do not commingle, in the same way, must not he [the dialectician] possess some science [*epistemes*] and proceed by the processes of reason ... to show correctly which of the classes harmonize with which, and which reject one another, and also if he is to show whether there are some elements extending through all and holding them together so that they can co-comingle... . (Cited in Whitehead, 1948, p. 98. Brackets added.)

Here Whitehead is alluding to the perennial distinction between *ars demonstrandi* and *ars inveniendi*; between the (deductive) logic of drawing necessary conclusions from given premises via rules of valid inference, and a "wider logic" which combines, blends and thereby multiplies entities to synthesize new compound concepts. We shall return to the Platonic notion of the "comingling" or "intertwining" of concepts (*symploke eidon*) in §4.

- ***"Modern" (Leibnizian symbolic) roots***

As will be discussed in §6, what Whitehead means by non-metric, axiomatic-synthetic geometry was essentially what Gottfried Leibniz (1646–1716) was developing in the 1670's and 80's with his work on *Characteristica Geometrica* ("symbolic geometry") and highlighted in a later article given the title "General Inquiries on the Analysis of Concepts and Facts" (*Generales Inquisitiones de Analysi Notionum et Veritatum*) from c. 1686. Here Leibniz was attempting to develop a "geometry of relations without magnitudes, independent of figures, embedded in a complete axiomatic and endowed with an expressive symbolism".[18] This is what later – at the end of the 19th Century – would be called a General or Universal Algebra, i.e. a general theory of symbolic reasoning. And the first to write a comprehensive treatise in English on Universal Algebra was none other than A. N. Whitehead (1998).[19]

In these works, Leibniz presents many of the axioms put forward by Whitehead in **APG** and makes the self-same remark concerning the status of the symbolic system thus developed, i.e. that it is not fixed to any *determinate subject area*, but that "...whenever these laws are observed, the present calculus can be applied".[20] Whitehead was fully aware of this:

> "The founder of the *modern point of view* … was Leibniz, who, however, was so far in advance of contemporary thought that his ideas remained neglected and undeveloped until recently; … especially "*Generales Inquisitiones de Analysi Notionum et Veritatum*". (Whitehead, 1948, p. 207. Emphasis added)

The crux of this "modern point of view" involves the realization that operative symbolic systems can be employed as heuristic tools for modelling the deep (unseen) structure of empirical targets which conform to their axiomatic rules. In this way, the symbolic system – or model – *itself* becomes an object of "empirical study".[21] This is what Leibniz called *cognitio symbolica*, and which today goes by such various names as surrogate reasoning[22], generative symbolism[23] and operative symbolic systems.[24]

Whitehead was also heavily influenced by – and made a thorough study of – Hermann Grassmann who, in turn, was well acquainted with Leibniz's earlier work on non-metric geometry (*analysis situs*) and was engaged in the very same "project". Indeed, he considered his *Ausdehnungslehre* (1844) to be "the natural development of Leibniz's ideas".[25] Their common aim was to develop a heuristic symbolic system that went beyond mathematics as such:

> "The more general aims of both Leibniz and Grassmann were the same – the setting up of a convenient calculus or art of manipulating signs by fixed rules, and of deducing therefrom true propositions for the things represented by the signs... In each case their geometrical calculus was a particular application to geometry of a wider calculus, for which each desired more than mere applicability to mathematics." (Heath, 1917, p. 38)

The "wider" epistemological issue here is that science develops not only in the light of new empirical discoveries, or the accumulation of empirical anomalies, or technical innovations. Scientific revolutions are also facilitated by way of "rational discoveries": i.e. anomalies and surprises emerging *from within* the very heuristic symbolic systems – or *models* – developed in order to represent empirical phenomena. This is the core idea behind what is today called Model-Based Reasoning (MBR) and model-based science.[26]

Thus the development of a general *science of symbols* (*Formenlehre/Morphologie*) goes beyond both mathematics and formal logic into a pure calculus of concepts, further facilitating heuristic reasoning and supporting the art of discovery. However, what I wish to show is that *ars inveniendi* is not only supported via abstract symbolic systems, but is also inherent in pre-symbolic conceptual modelling invented in ancient Greek science.

Although non-symbolic conceptual modelling cannot match operative symbolic systems in either demonstration or discovery in the areas of mathematics and natural science, it can be applied to other important areas that operative symbolic systems cannot (yet?) reliably represent in any "depth", e.g. in the so-called *deliberative arts*[27]. This includes such areas as futures studies and scenario development, policy analysis, strategy management, organisational design and statecraft. Indeed, the primary aim of this book is to show (1) how these "deliberative arts" are being treated today with non-symbolic, constraint-based combinatorial modelling and (2) how this type of heuristic modelling has a long history, co-developing with the very genesis of rational science.

1.2 Epistemological history

I have chosen to begin this Introduction with Whitehead's axiomatic treatment of non-metric geometry in order to emphasise how late 19th and early 20th century mathematics and philosophy of science were still in the process of rediscovering and further developing fundamental concepts emerging out of the 17th Century symbolic revolution. The study of this type of historical conceptual development belongs to what is called *Épistémologie Historique* – usually inverted in English as *Epistemological History*.[28]

The continental tradition of Epistemological History developed in the early half of the 20th century in France. It was not aimed simply at an "objective" chronological description of the history of theories of knowledge, but as an "evaluative" study of the history of scientific *concepts* developing within evolving epistemological-ontological frameworks – i.e. the *reciprocal development* of empirical research and rational foundations. Among other things, this involves what has been called *regulated* or *controlled anachronism*: a self-conscious approach by which historians of science "acknowledge anachronism as part of their awareness of the relationship between their own present and the past that they are studying"[29]; and by which to "uncover the contextual origin of *supposedly obvious* [modern] concepts, bringing to light the assumptions on which they rest".[30] This is expressed admirably by one of the great conceptual innovators and philosophers of science of the 20th century, Erwin Schrödinger (1887-1961) in "Nature and the Greeks" (1954). In studying ancient Greek science:

> "There is not only ... the hope of unearthing obliterated wisdom, but also of discovering inveterate error at the source, where it is easier to recognize. By the serious attempt to put ourselves back into the intellectual situation of the ancient thinkers, far less experienced as regards the actual behaviour of nature, but also very often much less biased, we may regain from them their freedom of thought." (Schrödinger, 1954, p. 18f)

Indeed, this is exemplified by three of the central figures in the 20th century "quantum revolution" – Schrödinger, Werner Heisenberg and Wolfgang Pauli – all of whom, by their own admission, looked "back" to Plato and the pre-Socratic philosophers for inspiration in formulating a theoretical framework for the interpretation of quantum physics.[31]

Epistemological history thus has a strong pedagogic-heuristic aspect. As has been pointed out repeatedly by prominent scientists and philosophers of science, scientific discovery is best facilitated by experiencing examples of earlier creative processes.[32] Writes Leibniz more that 300 years ago:

> "It is an extremely useful thing to have knowledge of the true origins of memorable discoveries, especially those that have been found not by accident but by dint of meditation [i.e. deliberative reasoning]. It is not so much that thereby history may attribute to each man his own discoveries, … [but] that the art of making discoveries should be promoted by considering noteworthy examples of it."(Leibniz, 2008, p. 22)[33]

1.3 Non-quantified modal modelling

At this point it is appropriate to provide at least a minimal definition of what is meant – in this work – by both the notion of a scientific *model* in general, and a *non-quantified model* in particular. As has been observed elsewhere, the notion of a scientific model, like the notion of a "system" or "theory", belongs to a class of concepts which essentially have unbounded domains. [34] The open-ended, emergent nature of these concepts makes it difficult to give them both an all inclusive (complete) *and* precise (internally consistent) definition. However, from the point of view of a *general theory of modelling*[35], we can at least supply a minimal, operative definition which suffices for the purposes of this study. We define a ***model*** (and we now drop the "scientific" qualifier and ask that it be understood) by way of the so-called E-R ("Entity-Relationship") schema, in which the essence of a "model" lies in its *entity-relationship structure*.[36] A model consists of:

A) Two or more (mental) constructs that can serve as variables which can support a range of states or values – otherwise called the variables' respective *domains*. We shall call these variables the *parameters* of the model.[37]

B) The establishment of relationships (e.g. mathematical/functional, probabilistic, logical) between the domains of the different parameters, such that each parameter is "connected to" (i.e. constrained by) at least one of the other parameters.

The development of these two components (variables and connective links) into a model is essentially an iterative process of (compositional) analysis and synthesis. In the analysis phase, variables and their respective domains are formulated which represent the model's

initial *problem space*. In the synthesis phase, connective relationships between parameters are defined and applied, thus *binding* the modelling space, determining its topological properties and constraining it in order to produce a solution or outcome space.

Thus the basic framework for a model is an internally connected, n-dimensional conceptual space which goes under a number of different names depending on the nature of the model, its area of application and the *properties of the space to be emphasised*: for instance, coordinate space, parameter space, configuration space, state space, phase space, or morphospace.

For the purposes of this work, a *non-quantified* model consists of:

A') A set of category (a.k.a. categorical) variables, the *discrete domains* of which consist of elements which are either *non-ordinal* or can be *rank-ordered*, but which are not associated with magnitudes or metric relationships. (In statistical analysis these are called *nominal* and *ordinal* scales respectively).

B') Relationships (or links) between the domains of the given variables are also established on the basis of non-metric (formal logical and/or *modal* alethic and deontic) constraints.

Now, such (discrete variable) combinatorial models or systems are not simply poor-man versions of continuous models or systems. They are characterized by what is called a *finite state space*, i.e. a discrete n-dimensional space representing the set of all possible states or configurations of the system being modeled. As such, they are subject to a number of discrete mathematical sub-disciplines, including the many varied facets of combinatorics, graph theory, matrix analysis, lattice theory and category theory.[38] In Operations Research and in AI, these two iterative steps are known as *formal concept analysis* and *constraint satisfaction*.[39] Furthermore, when *alethic* (possibility, necessity and impossibility) and/or *deontic* (prescription and proscription) constraints are systematically employed, this is an example of *modal modelling*, a species of modelling which has gained considerable interest during the past decade and has its own epistemological issues to deal with.[40] This will be further discussed in §3.

To summarise the organisation of this book: Chapters 2–3 present a theoretical and methodological background to chiastic cross-classification and "cross-order" modelling; Chapters 4-7 present how this form of modelling has developed historically in the works of Plato, Ramon Llull, Gottfried Leibniz and Fritz Zwicky. Finally, in Chapters 8–9 the modern theory of computer-aided constraint-based combinatoric modelling (a.k.a. General Morphological Analysis) is presented, with a number of applications in design theory, policy analysis, operations research and decision science.

However, the first question to be asked is: When did the process of cross-ordering concepts – in order to produce new complex concepts – begin, and why? All that may be conjectured at this point is that it must have been an integral aspect of the development of human cognition and spoken language itself, taking place in the preverbal "fog of prehistory".

2. In the Beginning was X: From Rhetorical to Epistemic Chiasmus

> "When chiastic structures are understood as more than stylistic devices, endless possibilities ... are revealed. It appears that chiasmus can indeed be used for both the production and discovery of meaning..."[41]

X (χεῖ, Latin: *chi* – pronounced as "key") is the twenty-second letter of the ancient Greek alphabet. Besides being a standard consonant, because of its X-form it has also come to be associated with – or be a generalised symbol for – a cross-order pattern of inverted parallelism. For this reason, such cross-order patterns were given the name *chiasmus*, which is the Latinised form of the classical Greek *khiasmos* and the verb *khiazein*, which literally means "to mark with the letter ***X***". The Latinized term, however, meant more like "to arrange or order in the form of ***X***". In this signification, what *chiasmus* actually denotes predates the letter ***X***, predates the Greek alphabet and predates written language altogether. Chiasmus is a fundamental aspect of human reasoning. It stands for any "cross-order" arrangement of objects, lines, symbols, diagrams or any type of locution: words, concepts, expressions, phrases or whole narrative accounts. In this sense, it is the *minimal unit* – the "atom" – of cross-classification and conceptual integration.

Chiastic structure can be divided into three broad, overlapping categories: iconic, rhetorical/literary and epistemic. Since the main purpose of this work is to demonstrate how certain forms of epistemic chiasmus can serve as a basic framework for combinatorial heuristic modelling, purely iconic chiasmus is less relevant here.[42] It is, however, interesting to begin by briefly considering its *rhetorical/literary* forms, because these also have a heuristic function and partially overlap pure *epistemic* chiastic forms.

2.1 Elementary examples of rhetorical chiasmus

Rhetorical chiasmus – along with metre, rhyme, formulaic expressions and other rhetorical "figures" – has served both as a mnemonic device for easier learning and to help preserve narrative structure over the generations of oral tradition.[43] It emerges from the fog of prehistory and makes its transition into "literacy" in the early written works based on such traditional narratives.[44] However, there is more going on here than the preservation and transmission of oral-based knowledge. Rhetorical chiastic structure has at least a proto-epistemic faculty: It both contrasts and integrates – combining, permuting and/or negating concepts – thus creating more complex relationships and deeper meaning: i.e. it facilitates *inventio* through conceptual combination and integration.[45]

Maurice Merleau-Ponty (1968) associates rhetorical chiasmus with the *intertwining of concepts*[46] – recalling Plato's notion of *symploke eidon* as creative concept conjunction (see §4, *infra*). Wiseman (2019) likewise notes that chiasmus acts as:

> ".... a pattern of thought, and organizing schema, a structure that determines ... form and content" [... and] "above all *inventio*, the invention of subject matter and the logical arguments that give form to it." (p. 87)

Like most everything associated with combinatorial structure, chiastic arrangement can be "folded" into any degree of complexity – e.g. in structural variations called diamonds, hour-glasses, hexagons, octagons, chains, rings and "argyle".[47] However, here we are mainly concerned with its basic principle as this is expressed in its *minimal unit* of cross-order structure: the simple "**X**"-form. For, even in its minimal ("atomic") form, we find different "modes of chiasma" expressing different logical-epistemic structures. These different modes are based on combinations of four logical relationships: (1) *bisociation*, (2) *contrariety*, (3) *contradiction* (or logical *complement*, where "¬" denotes negation) and (4) *logical subordination* (Table 2.1).

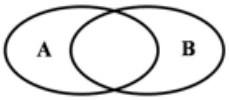
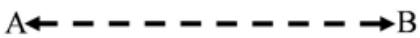
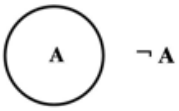
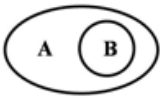
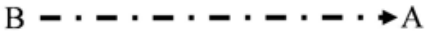
Concept relationships	Logical representation	Chiastic representation
Bisociation: Independent concepts that can be true simultaneously and conjoined. **(A and B)**		
Contrariety: Two concepts that cannot be true simultaneously, but can both be false. **(A or B)**		
Contradiction: If one of the concepts is true, the other must be false, and *vice versa*. **(A or not-A)**		
Logical subordination: When the truth of a subordinate concept implies the truth of its superordinate. **(B ⇒ A)**		

Table 2.1: Four logical concept relationships

At present there seems to be no generally accepted standard terminology for all of the variants of even the simplest chiastic forms. One must therefore either accept or alter currently used terms, and create new ones where needed. Except for "Antimetaboles", the descriptive terminology used here is my own *ad interim* concoction. It will have to do for present purposes. Thus the four modes of basic chiastic structures treated here (Fig. 2.1, *infra*) are:

- Antimetaboles (not applicable to Table 2.1.)
- Bisociate-contraries
- Bisociate-contradictories
- Contrary-contradictories

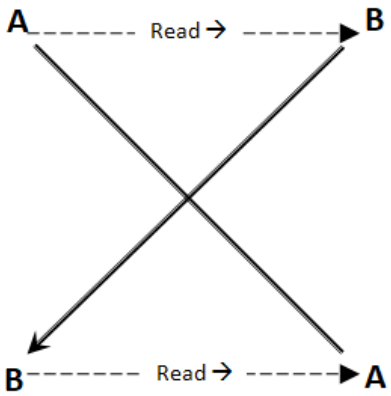


Fig. 2.1a: Antimetabole

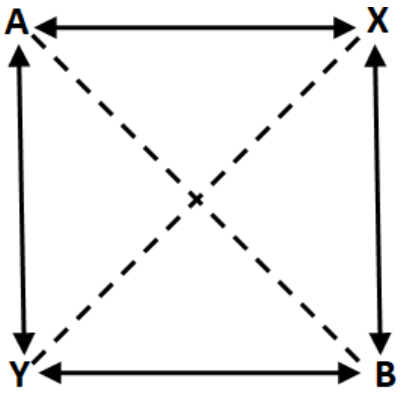


Fig. 2.1b: Bisociate-contraries

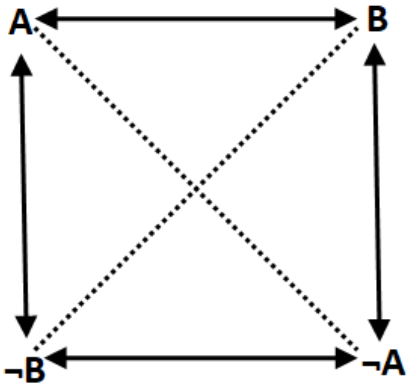


Fig. 2.1c: Bisociate-contradictories

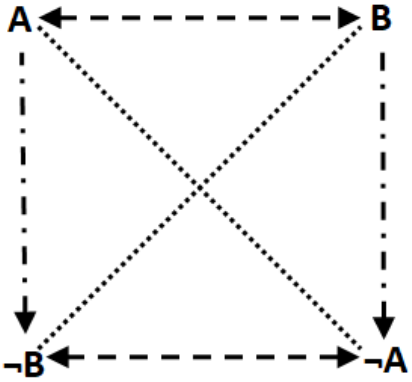


Fig. 2.1d: Contrary-contradictories

Figure 2.1 Four basic chiastic structures

2.1.1 Antimetabole (Fig. 2.1a)

2.1.2 Chiastic form: AB::BA and its "corrective" variant AB::¬(BA)

In this special form of rhetorical chiasmus (some do not considered it chiasmus *proper*) the A and B elements are inverted and repeated exactly.

- "One for all and all for one". (Popularized by Alexandre Dumas[48])
- "[You should] Do unto others as you would have others do unto you". (Universal)

The reciprocal relationships expressed in such antimetaboles open up a wholeness which transcends the individual *cola* (the terms on either side of the colons) and creates a connection between the part and the whole; the particular and the universal.

The *negation* of one or the other *cola* expresses a kind of antithesis called *corrective or refutative* antimetabole.[49] For example:

- "Eat to live, *don't* live to eat" (Attributed to Socrates[50])

- "Ask *not* what your country can do for you; ask what you can do for your country (J. F. Kennedy – one of the most famous of popular antimetaboles)

Now, to show how even the seemingly simplest form of chiasmus can have the potential for generating formal epistemic structure, consider the following *corrective antimetabole*:

- "*All* stories are narratives, but *not all* narratives are stories".[51]

Here we see an antimetabole clearly expressing a logical relationship, based on the presence of the two logical operators "all" and "not". It indicates that "story" is an asymmetric, subordinate concept (or subclass) of the more inclusive concept of "narrative". This may seem somewhat trivial at this point but, as we shall see, such rhetorical figures contain the seeds of complex logical (syntactic) structure.

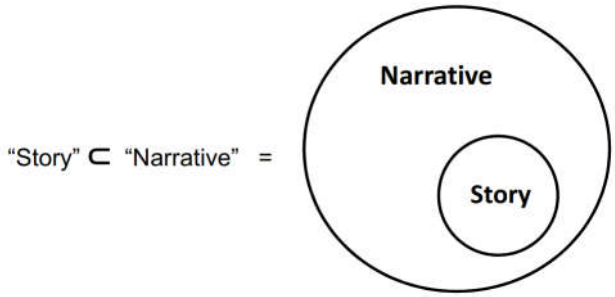


Fig. 2.2: "Story" as sub-type of "narrative"

2.2 Bisociation and "chiastic squares"

Here we get into chiasmus *proper*, which is more diverse and complex than "inverted parallelism" *per se*, involving combinatoric cross-classification and conceptual integration. The two classical structures are *bisociate contraries* (Fig. 2.1b) and *bisociate contradictories* (Fig. 2.1c). The term *bisociation* was introduced by Arthur Koestler in his 1964 book "Act of Creation" to denote creative concept conjunction which results when two (seemingly) unrelated contexts or patterns of thought are combined to produce more complex concepts or patterns.

> "The creative act is not an act of creation in the sense of the Old Testament. It does not create something out of nothing; it uncovers, selects, re-shuffles, combines, synthesizes already existing facts, ideas, faculties, skills. The more familiar the parts, the more striking the new whole."[52]

Bisociation is a handy concept which has inspired further research in the study of creativity.[53] However, as Koestler was well aware, it was nothing essentially new. It is a modern expression of the notion of conceptual blending dating back to Plato and the early

Neoplatonic scholars (see §4), and to Leibniz's notion of combinatorial *ars inveniendi* (see §6). We use the term here to denote both the *operation* and the *product* of combining two or more co-ordinate concepts to produce a complex concept which "contains" the properties of both of the simpler ones, and which can also create new conceptual entities which transcend both. (In extension, this "operation" can also apply to variable domains or sets of concepts, and even to entire "frames of reference".)

2.2.1 Bisociate contraries (Fig 2.1b) Chiastic form: AX::BY and AY::BX

Whereas antimetabole is based on (*permutational*) reversal between the same two concepts, bisociate conjunction is *combinatorial* and based on the cross-ordering of two pairs of contrary concepts. This is a canonical form of chiasmus that opens up an infinite universe of inventive conceptual blending.

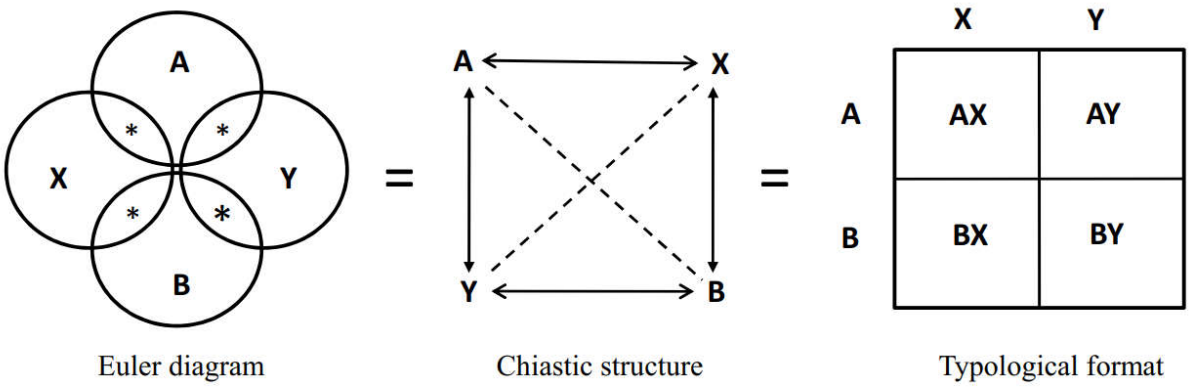


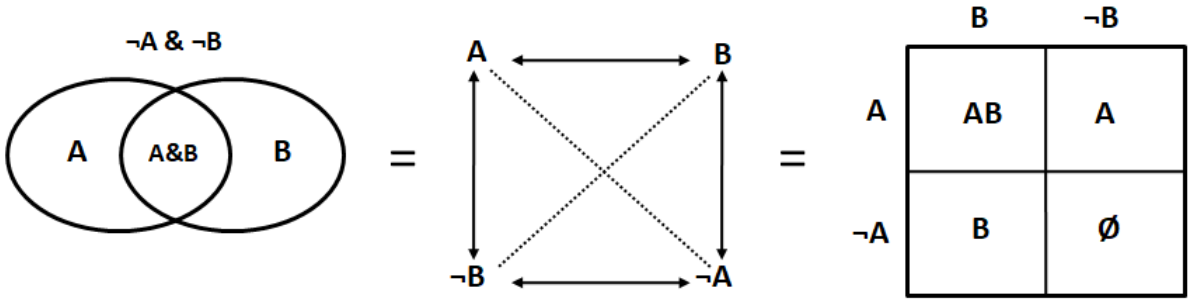
Fig. 2.3: Two bisociate contraries (A or B) & (X or Y) cross-ordered

In this form, two *contrary (disjoint) terms* [(A or B) & (X or Y)] are cross-ordered to produce a four-fold classification. Here are two examples which help us to identify "reversals in perspective or behavior"[54] , inviting us to take a stance about what is important in life:

- "I'd rather **die** on my **feet** than **live** on my **knees**." (Attributed to Emiliano Zapata[55]).
- "Better **poor** and **healthy** than **rich** and **sick**. (The sardonic "no-brainer"-version being: "Better **rich** and **healthy** than **poor** and **sick"**)

2.2.2 Bisociate contradictories (Fig. 2.1c) Chiastic form: AB::¬A¬B & A¬B::B¬A

This form of chiasmus pits a single bisociate pair of concepts (**A** & **B**) against its respective negations (¬) or "logical complements". As above, this conjunction produces a 2x2 ("four-fold") structure producing four possible combinations (Fig. 2.4.)



Euler diagram Chiastic structure Typological format

Fig. 2.4: Bisociate terms (A&B) crossed with their logical complements

- "Sir, your manuscript is both **good** and **original**; but the parts that are **good** are *not* **original**, and the parts that are **original** are *not* **good**." (Attributed to Samuel Johnson[56])

A further example – combining two chiastic modes and called a Confucian antimetabole[57] – is a bit more complex, having the form **A¬B** is **C::B¬A** is **D:**

- "**Theory** without **practice** is *empty*; **practice** without **theory** is *blind*." (Some variant of this has been attributed to Gottfried Leibniz, Immanuel Kant, Karl Marx and Albert Einstein – take your pick.)

Now, considering their respective "typological" representations, *bisociate contraries* and *bisociate contradictories* seem to have quite similar structures, the difference being that the former deals with two (crossed) bisociate pairs of contrary concepts, while the latter deals with a single bisociate pair crossed with its respective negations. However, the role of *negation* is crucial here, creating the potential for more complex relationships with a single pair of bisociate concepts.[58]. With the fourth mode – *contrary contradictories* – we are on our way towards pure formal-epistemic chiasmus.

2.3 Epistemic squares of opposition: Semantic and Syntactic

The type of chiastic structures that we have labeled *contrary contradictories* (Fig. 2.1d) are conventionally called "Squares of opposition". I agree with those who feel that this designation is a bit too general, since *bisociate* forms of chiasmus – as described above – also involve "oppositions". However, it is not inconsistent with the basic point to be made of such squares, i.e. that they chiastically combine both *contrary* and *contradictory* oppositional relationships, and contain no conjunctive (bisociate) relationships. In any event, even if *contrary contradictories* may be more technically descriptive, "Squares of opposition" is the generally accepted term, which we shall now follow.

There are two different "modes" of such squares. One mode works at the *semantic* level and is called the Semiotic square. It relates two contrary natural-language concepts to each others' negations. The other mode works at the *syntactic* level and is called the Logical square. It pits two contrary (logical or modal) "operators" against their respective negations.

Now, although the Logical square of opposition predates the "modern" Semiotic square by more than 2000 years, we start with the latter because – in a sense – it is epistemologically prior. By this I mean that the Semiotic square treats relations of *meaning* between concepts, whereas the Logical square is a further abstraction that treats *relationships between formal relationships*. For our purposes it is the Logical square that is central, as its various different *modal* versions make up the basis for the constraining rules involved in constraint-based combinatoric modelling. However, leaving the Semiotic square out altogether would be delinquent, so we review it first "for the record".

2.3.1 The semiotic (semantic) square of opposition

The Semiotic square of opposition was introduced in *Sémantique Structurale* in 1966 by Algirdas Greimas (1917–1992).[59] It was an extension and formalisation of the work of the 20th century structuralists such as Roman Jakobson, Vladimir Propp and Claude Lévi-Strauss.[60] As a framework for modelling the relationships between *semiotic signs*, it works with natural language concepts and can be applied to virtually any contrary-like terms. Thus although it is based on – and isomorphic with – the traditional Logical square (see §2.3.2 below), it is *not* concerned with "… logical or mathematical constructions, which are independent as formulations of 'pure syntax'…"[61], but strives to elucidate *meaning and signification*.[62]

The main feature of the square is to discover or create new *meaning* by forcing one to confront the partial disparity between contrary and complementary expressions. For instance: when one starts out with semantic contrariety S_1 – S_2 (Figure 2.5), the square immediately invites us (provokes us) to complete the cross-order by finding examples – or the closest possible meanings – of their complements (negations) at $\sim S_2$ – $\sim S_1$ in the lower diagonal corners. However, these complements must also make sense as *implications* involved in the asymmetric relationships between S_1 – $\sim S_2$ and S_2 – $\sim S_1$. Depending on the nature of the contraries on the upper level, relationships can become "semantically complex" – to the point of phantasmagorical – which is part of its allure.

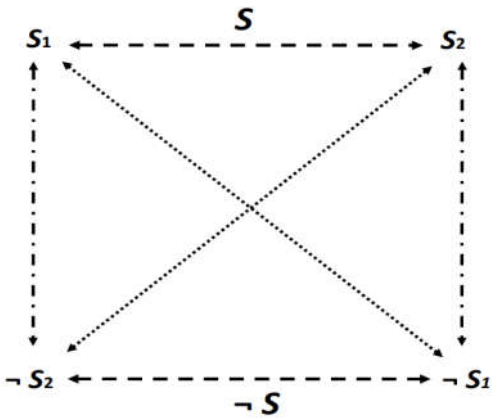


Figure 2.5: Generalised Semiotic Square

Two examples can demonstrate this mixture of contrast, combination and implication:[63]

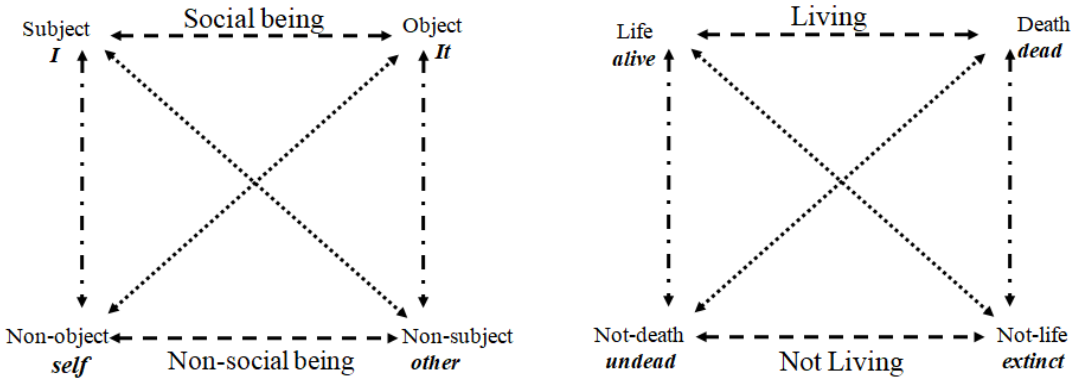


Fig. 2.6: Two Semiotic squares: "Social Being" and "Living"

2.3.2 Logical Squares of opposition (Contrary complements)

What is meant here by the term "pure epistemic chiasmus" is a purely formal (syntactic) framework that has no determined semantic content until such is provided, again echoing Leibniz's (and Whitehead's) notion of purely syntactic systems that can be applied to any subject that abides by its formal rules. Thus "logical squares of opposition" go beyond verbal discourse to treat relationships not between "things" (or names of things), but are diagrammatic representations of how negations interact with certain logical (including modal) *operators*. They treat *formal relationships* between *formal relationships*.[64]

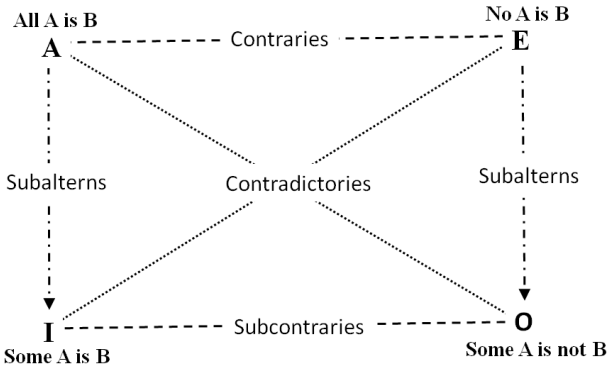


Fig. 2.7a: Square of Opposition: Chiastic structure.

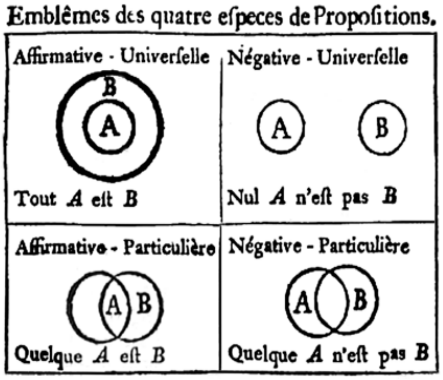


Fig. 2.7b: Euler's rendering as four-fold table

Fig. 2:7: Logical Square of Opposition: Chiastic and Euler's typological format.

The roots of the logical square of opposition are found in Plato's Sophist[65] (§ 4, infra) but its first formal *textual* description was broached by Aristotle in *De interpretatione*. There, he made the explicit distinction between *contrary* and *contradictory* relationships occurring between four basic *categorical propositions* (**A**, **E**, **I** & **O** in Fig. 2.7a). There is no evidence that Aristotle produced any diagrammatic representation of this. Its first occurrence as a

completed *heuristic diagram* is usually ascribed to Apuleius of Madaura (c.124–c.170). By the time of Boethius (c. 480–524) the diagram was complete with its six connections based on two contrary, two contradictory and two subaltern relationships (Fig. 2.7a).[66]

The four corners of this traditional Logical Square represent the four basic *categorical propositions* recognized in syllogistic logic:

A *Universal affirmative*: All S is P
E *Universal negation*: No S is P
I *Particular affirmative*: Some S is P
O *Particular negation:* Some S is not P

The logical relationships within this and other forms of Squares of Opposition are:

- Two propositions are *contraries* if they cannot both be true but can both be false.
- Two propositions are *subcontraries* if they cannot both be false but can both be true.
- Two propositions are *contradictory* if they cannot both be true *and* cannot both be false.
- A proposition is *subaltern* of another if it must be true if its *superaltern* is true, and the *superaltern* must be false if the *subaltern* is false.

This traditional "Aristotelian" square of opposition is also referred to as the *categorical* or *assertoric* square because it makes categorical or unconditional *assertions*. This is the basis for the *syllogistic* – the original discipline of formal (deductive) logic. It is concerned with how the four categorical propositions (**A**, **E**, **I** & **O)** can be combined in order to produce *valid inference* and thereby "logically consistent" judgments (see §6 for a more detailed account). However, this is only one of several types of logical reasoning. There are other important types of "logical relationships" that are based on the same *contrary-contradictory* framework as the classical *assertoric* logical square. These are called *modal logics*.

2.3.3 Modal logic and modal constraints

Unfortunately, there is a terminological ambiguity involved here as concerns what should be included under the rubric of "modal".

> Narrowly construed, modal logic studies reasoning that involves the use of the expressions 'necessarily' and 'possibly'. However, the term 'modal logic' is used more broadly to cover a family of logics with similar rules and a variety of different symbols.[67]

Thus, using the "narrow" sense of the term, what is called *alethic* logic, concerning possibility and impossibility, is "modal", whereas *deontic* logic, the normative logic of prescription and proscription, is to be considered a separate family. However, here we use the "broader designation" which involves both alethic and deontic logic. Indeed, even traditional *assertoric* logic can be considered a particular "mode" of logic. (There are other members in the family of "modal logics", including *temporal*, *epistemic* and *doxastic*, which we do not treat here.)

The two most prominent examples of *modal logic* – which also happened to be those that are most relevant for constraint-based combinatorial modelling – are *alethic* and *deontic*, and these have their own corresponding "logical squares" (Fig. 2.8).

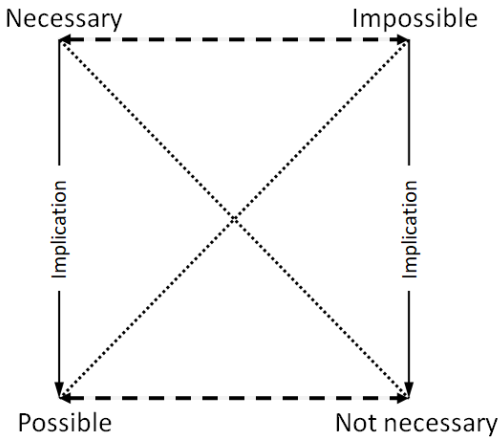


Fig . 2.8a Alethic square of opposition

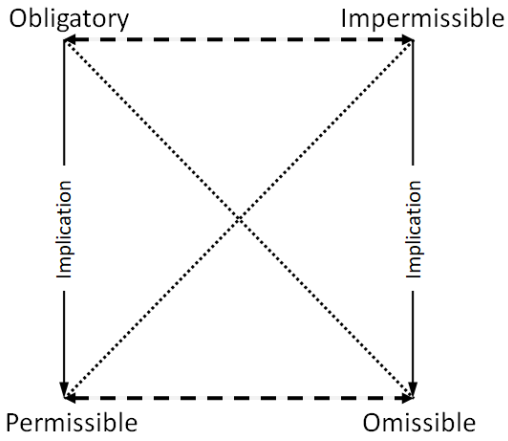


Fig . 2.8b Deontic square of opposition

Fig. 2.8: Alethic and Deontic Squares of Opposition

For our purposes, the main take-home message is that these three "modes of logic" represent the primary types of constraining rules which produce "inference by exclusion" in combinatorial heuristic modelling: the *assertoric* concerns (internal) logical consistency; the *alethic* concerns (external) nomological-empirical possibility; and the *deontic* concerns normative permissibility.

There is one last chiastic square to explore before we can begin to apply all of this to the history of modelling theory. And here we find a remarkable similarity between the "Logical square opposition" described above, and the foundational structure of classification theory.

3. The Chiastic "Square of Classification"

> "One of the main problems for a general theory of classifications is that classifications do not amount [solely] to *hierarchical* classifications. There are in fact many other kinds of classification ... [which] may have the same subjects inserted in many classes, which obviously means that *classes intersect*." (Parrochia & Neuville [68])

> "Intersection is heuristic, and progress is interwoven." *(*Michel Serres[69])

3.1 Classifying classification

Classification, as the faculty to discern, arrange, group and interrelate objects, ideas and concepts, is ubiquitous (and essential) to the human condition.[70] Classification *theory***,** on the other hand, is the self-conscious investigation, development and application of methods of classification. It may be argued that the development of a coherent theory of classification is a prerequisite – or co-requisite – for the development of a rational science, including at least a prototype theory of modelling. Nowhere is this more evident than in Plato's "theory of ideas" (see §4, infra).

However, before going into the ancient origins of classification theory, we first need to look at how modern classification theory is organized, in order to better understand its epistemological history. Modern classification theory has been broken down into three levels:[71] 1) *First-order theory* concerns methods of classificatory construction for *different areas of use*; 2) *Second-order theory* concerns how to maintain, update, mix and in general to work with *already developed systems*; and 3) the *Foundational theory* of classification concerns its epistemological and definitional basis, including its *structural means of representation*. It is at this level that we can relate classification theory to modelling theory.

At its *foundations*, classification theory has its own built-in chiastic order based on two structural properties:[72]

1. A classification system can either be *hierarchical* or not (involve class *subordination* or not).
2. A classification system can either include *overlapping classes* or not (allow for class *conjunction* or not).

On the basis of these two "dimensions" we can formulate a 2x2 *bisociate contradictory* cross-classification of classification, here shown in four different formats in order to emphasise the diverse modes of diagrammatic representation available (Figures. 3.1a - d).

Note the structural similarity between this Square of Classification – especially as rendered in Euler format (Fig. 3.1c) – and that of the Logical Square of opposition (Fig. 2.7b, *supra*). Both concern the formal relationships of subordination, disjunction and conjunction. The difference is that the Square of Classification is an example of *bisociate contradiction* whereas the Logical Square is an example of *contrary contradiction*.

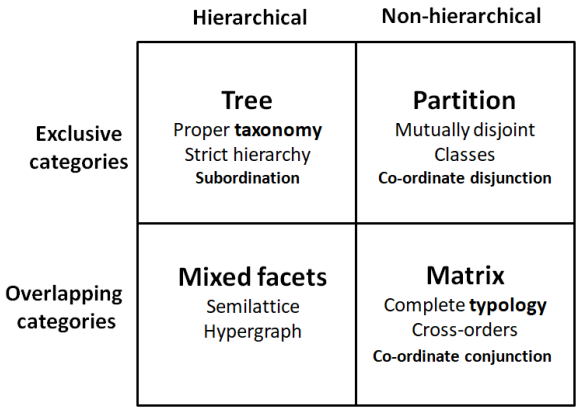


Fig. 3.1a: Verbal typology

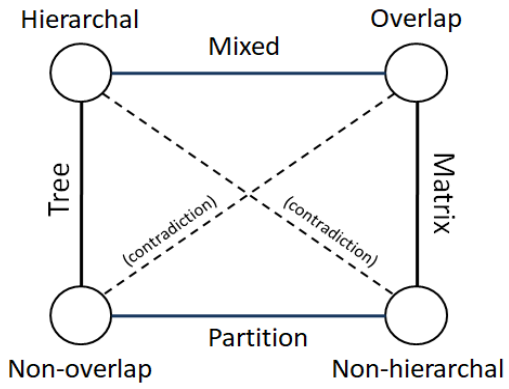


Fig. 3.1b: Chiastic structure
Bisociate contradictory: AB::¬B¬A

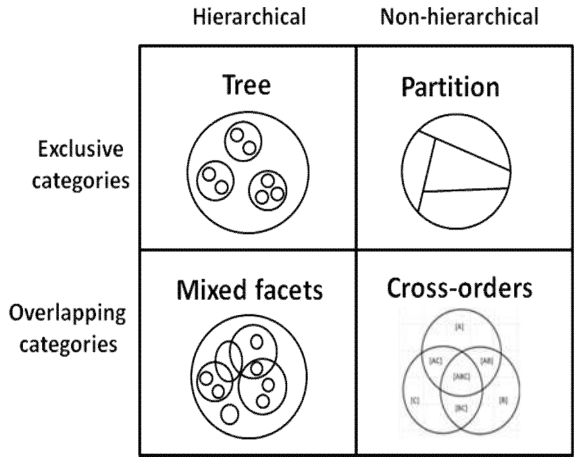


Fig. 3.1c: Euler-diagrams

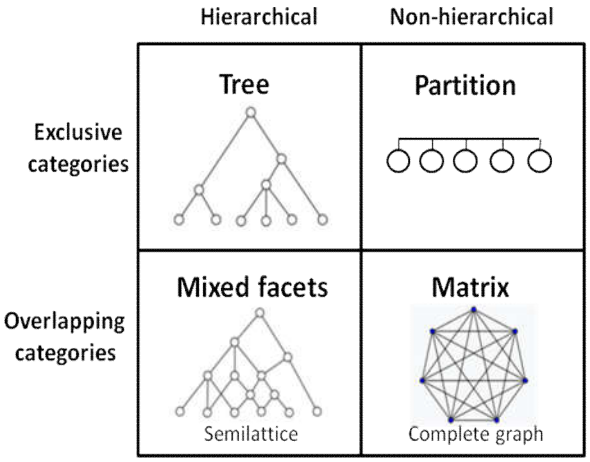


Fig. 3.1d: Graph structure

Fig. 3.1: Four diagrammatic representations for the 2x2 classification of classification

(I.) Partition: (Disjunctive coordinate)

"... a partition on a set U is a set of non-empty subsets (or 'blocks') of U where the blocks are mutually exclusive (no intersection among the distinct blocks) and jointly exhaustive (the union of the blocks is U).[73]

(II.) Tree: (Proper) Taxonomy (strict hierarchy; subordination)

A nested sequence of partitions, where there is only one *fundamentum divisionis* for each act of division. Subordinate species must be mutually exclusive and (ideally) disjunctively and exhaustive."[74]

(III.) Matrix: (Full) Typology (conjunctive coordinate; full cross-order)

A coordinate system of discrete variables or class attributes which takes the form of a combinatorial matrix orthogonally relating the attributes of the respective variables.

(IV.) Mixed facets: (multiple inheritance hierarchy) Combinations of II & III.

Note also the different aspects of the property "coordinate" in (I) and (III):

- (I) *Coordinate* in the sense of being on the same taxonomic level but *disjunctive.*
- (III) *Coordinate* in the sense of being on the same taxonomic level but *conjunctive,* as in a co-ordinate system of orthogonal dimensions.

However, the main historical-epistemological issue here is the structural distinction between hierarchal "trees" (II) and cross-order "matrices" (III), which represent two fundamentally different (albeit complementary) modes of classification.

3.2 Taxonomies and Typologies

In the 1930's, William Chase Greene and colleagues at the *Societas Philologica Americana* carried out a comprehensive study of logical and geometric (diagrammatic) symbolism in the Platonic *Scholia* – i.e. the 2000+ year old scholarly tradition of commentary, annotation and interpretation of the Platonic corpus.[75] Two of the most persistent and important of these diagrammatic structures fall into two natural categories: (genealogical) *trees* and (combinatorial) *matrices*. Figures 2.3a & 2.3b are examples of these two structures from the late medieval *Scholia*.[76]

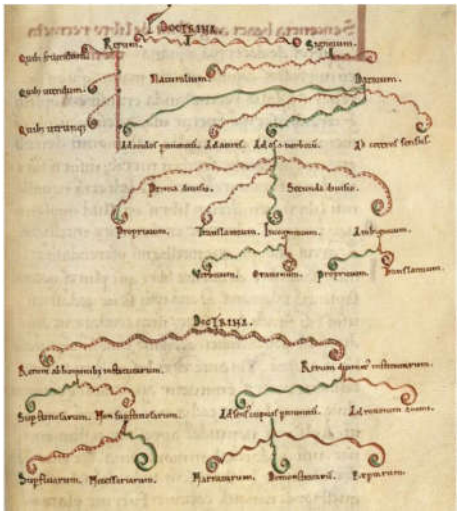


Figure 3.2a: Multi-level medieval tree diagram

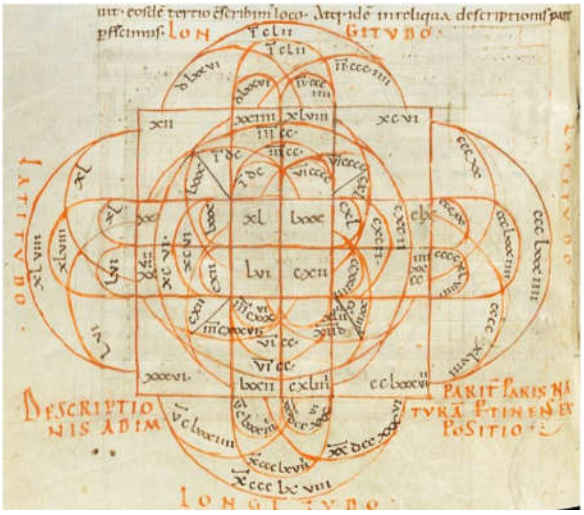


Figure. 3.2b: "Cross-order" matrix

Such diagrammatic classification schemes embody epistemic structure and mapping devises, both for knowledge *representation* and knowledge *generation*.

> "By manipulating and inspecting a diagram we can, due to the underlying conceptual mapping, draw inferences about the objects represented by the diagram (or rather: about the objects as they are conceptualised under the given metaphor or blend) ... diagrams are material anchors for conceptual mappings."[77]

As far back as Plato and his "Theory of Ideas", objects of knowledge were thought of as being defined by way of the spatial metaphor of "containment" – i.e. *inclusion within* and/or *in conjunction with* other concepts or properties. Indeed, the idea of being properly "defined"

was more or less synonymous with being properly "classified" within such a conceptual framework.[78] Furthermore, properly defined concepts are a prerequisite for discovering new types of relationships and new forms of knowledge.

> "Classification schemes not only *reflect* knowledge by being based on theory and displaying it in a useful way, ... but also classifications *in themselves* function as theories do, and serve a similar role in inquiry: that is, the role of explanation, parsimonious and elegant description, and the *generation of new knowledge*."[79]

Since we have hitherto devoted a good deal of text in distinguishing between *hierarchal* and *conjunctive* classification, we proceed by further defining this distinction. But first a warning:

For historical reasons, classification theory labours under an unfortunate ambiguity in terminology which has, on occasion, led to great confusion. The classical Greek word *taxis* means *order* or *arrangement*. On this basis, the term "taxonomy" has/is sometimes used generally to denote *any and all types* of classification schemes or structures. On the other hand, *taxonomy* is also used in a more specific sense (*inter alia* in biological classification schemes and evolutionary theory) to denote a hierarchal structure of linear inheritance (genealogy), represented by a *tree structure* (what we have earlier termed Linear-Hierarchal Classification - LHC). In this context, a pure (or "proper") *taxonomy* has the property of *strict hierarchal class inclusion* with no overlapping categories. This distinguishes it from a (proper) *typology* which has a non-hierarchal "coordinate" structure and consists *exclusively* of overlapping categories. This ambiguity between the two meanings of the term *taxonomy* has led to a number of futile controversies, one of which concerns the relationship between taxonomic structure and deductive reasoning (see below).

However, in contemporary *classification theory and research*, the difference between taxonomic and typological structure is important, even if the specific terms themselves are not always applied consistently. And although these two structures can certainly be combined, for instance in Plato's method of divisions (see §4) and in modern *faceted* information retrieval systems[80], we nonetheless want to clearly distinguish between their "proper" notions, since their formal structures represent two quite different (albeit complementary) forms of reasoning – which is exactly what we wish to highlight.

3.2.1 Taxonomy

A (proper) *taxonomy* is a system of classification that categorizes phenomena into mutually exclusive and exhaustive sets with a series of discrete, hierarchically *nested* decision rules of linear inheritance. It consists of a strict hierarchal tree-structure where only *inclusion* and *disjunction* are allowed. Except for the "highest" category (*summum genus*), each category is entirely included within at least one higher-order category. Such a *strict* species/differentia hierarchy allows for no overlapping (intersecting) classes or concepts. Figure 3.3 shows three ways to visually represent a proper taxonomic structure. As can be seen most clearly in the *spatial logic* of the so-called Euler diagram[81] (Figure 3.3c), the only types of logical relations allowed are *class inclusion* and *class disjunction*.[82]

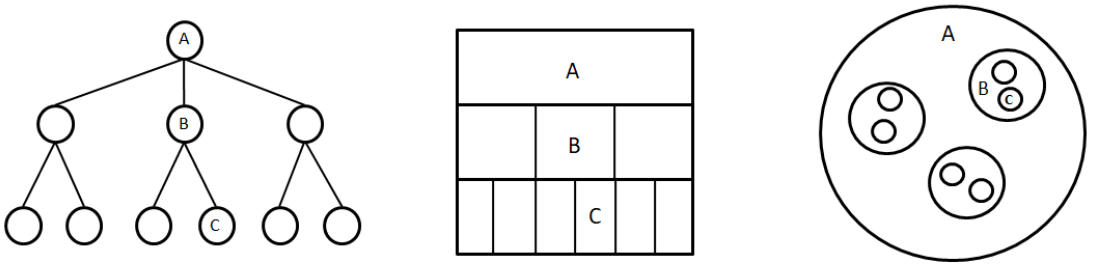


Fig. 3.3a
Tree format

Fig. 3.3b
Block format

Fig. 3.3c
"Euler" diagram

Figure 3.3: Three representations of a proper taxonomic hierarchy

Note that proper taxonomies have both *demonstrative* and *inventive* features. On the one hand, *class inclusion* as such is a relation expressing *logical entailment*. Thus, *once formed*, proper taxonomies express (and visualise) sequential-deductive inference.[83] On the other hand, the very process of developing such a hierarchal *species-differentia* structure is a (compositional) analytic procedure of definition by classification – and is thus potentially *inventive*.[84]

3.2.2 Typology

> "...perceiving the world and structuring it by means of types and typologies is an essential and intrinsic aspect of the basic orientation of actors to their situations. ...typologies are ubiquitous, both in everyday social life and in the language of the social sciences." [85]

A typology (a.k.a. multiple crossed-orders) is a *combinatorial classification* system based on multiple independent criteria. It consists of a set of category variables or attribute lists which are cross-ordered or *co-ordinated* (as in a Cartesian product), thus superimposing their domains upon one another. It identifies and inter-relates "simple" (one dimensional) concepts in order to create and explore the more complex multidimensional (multivariate) domain of concepts which can be compounded out of these simple concepts. This gives rise to a combinatoric space called a *typological field* or *type-space* which contains *only* overlapping (or conjunctive) concepts. Each specific formal "type" in the type-space consists of a unique (non-ordered) combination of its co-ordinate attributes.[86]

Now, once that the set of category variables involved are analytically defined, no decision rules are used in creating the initial type-space. Initially, there is no concern as to whether all its (extensional or intensional) conjunctions are actually possible or meaningful. It thus acts as *heuristic search space*.

Figure 3.4 shows a 2-species binary-variable typology in three different formats: Euler diagram, chiastic square and the ubiquitous "four-fold table". This is the *minimal unit* of a typological space, its simplest (canonical) form shown in Figure 3.4c. This also demonstrates a fundamental aspect of this type of cross-order modelling: namely, that the (extensional) intersection or (intentional) conjunction of two or more co-ordinate terms, concepts, variables or sets *automatically* creates a typological coordinate space.[87]

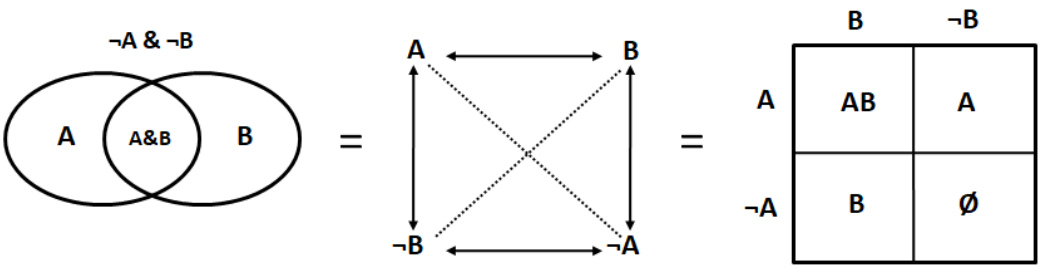


Fig. 3.4a
Euler Diagram

Fig. 3.4b
Chiastic Format

Fig. 3.4b
Typological Format

Figure 3.4. Three representations of a 2-species typology giving 4 types (n=2, k=2)

Now, we can immediately see how typologies can be extended in two ways: (1) by *domain extension*,[88] i.e. by extending the *number of elements* in one or more of its dimensions (i.e. category variables) (Figure 3.5); and/or (2) by *dimensional extension*, i.e. by increasing the actual number of dimensions of the field (Figure 3.6).

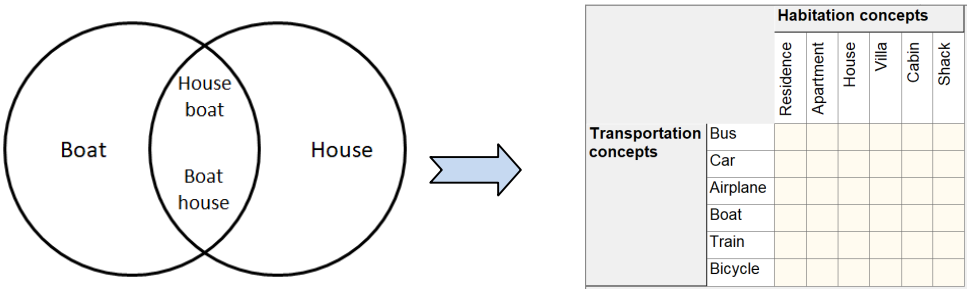


Figure 3.5: 2-dimentional binary-variable typology with extended domains.[89]

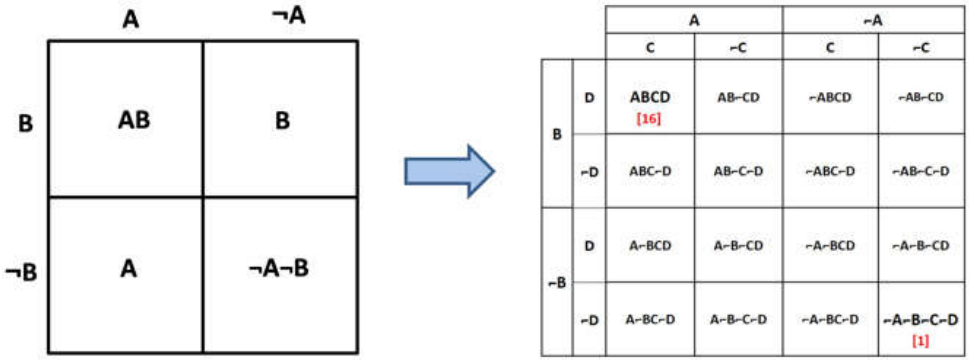


Figure 3.6: 2-dimentional binary-variable typology extended to 4-dimensions

Finally, in preparation for what is to come, we can see how a taxonomic "tree" is transformed into a typological "matrix". This is done by introducing *coordinate concepts* – instead of *sub-ordinate concepts* – into a proper taxonomic tree structure (Fig. 3.5). When two (or more) contrary concepts (A or B) can *each* be associated with two (or more) other contrary concepts

(X or Y), then we have what is called "parallel division", or a "quasi-tree" (Fig. 3.5a). This, in turn, can be represented as "cross-division" (Fig. 3.5b), its bisociate contrary *chiastic* representation, which then can be expressed as a cross-order matrix or typology (Fig. 3.5c).

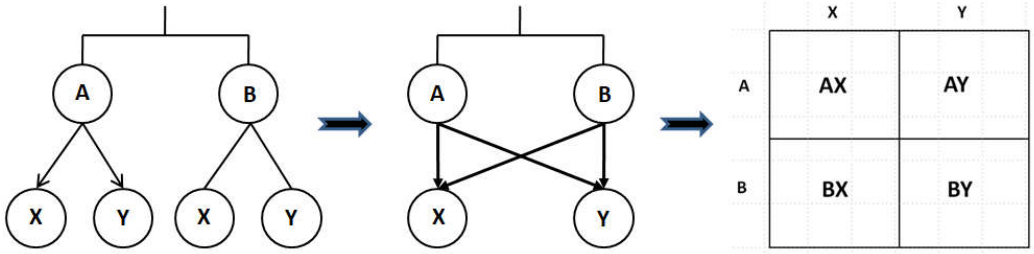


Fig. 3.7a
Parallel division

Fig. 3.7b
Cross division

Fig. 3.7c
Typology

Figure 3.7: From parallel division to typological matrix

3.3 Cross-classification and cross-cutting concepts

A beloved child has many names. By their very nature, many of the concepts discussed here are employed in conceptually analogous ways – but in different terms – in virtually every domain of science, for the simple reason that they are fundamental to all areas of science. This has been referred to as Cross-Cutting Concepts.[90] Consider, for instance, the following terms (hardly exhaustive), all of which refer to essentially the same general notion – the nature of the *communion or coordination of concepts* – but from different disciplinary points of view.

- Associative concepts
- Bisociation
- Block modelling
- Chiastic order
- Class intersection
- Combinatorial heuristics
- Combinatorial modeling
- Combinatorial synthesis
- Concept lattice
- Conceptual convergence
- Conceptual integration
- Conjunctive clustering
- Conjunctive coordination
- Conjunctive universals
- Constraint-based modeling
- Contingency table
- Coordinate concepts
- Cross table
- Cross-classification
- Cross-division
- Cross-order modelling
- Cross-space mapping
- Dimensional classification
- Double-scope blending
- Formal concept analysis
- Four-fold classification
- Four-fold table
- Latent class analysis
- Logical Space Diagrams
- Matrix modelling
- Morphospace
- Multi-label classification
- Multiple cross-orders
- Multiple inheritance
- Overlapping concepts
- Paradigmatic classification
- Parallel division
- Pivot table
- Polythetic classification
- Property space
- Pseudo-tree
- Quasi-hierarchy
- Semi-lattice
- Synthetic inference
- Typology
- Verbal matrix
- *Symploke eidon* (Plato)
- *Ars combinatoria* (Leibniz)

Table 3.1: Domain cross-cutting concepts concerning the "science of cross-classification"

These are presented only to emphasize the extent of the presence of these underlying epistemic concepts in modern science. (There seems to be no one "best way" to cluster them, so I gave up effort and list them alphabetically.)

All of these terms, in their different ways, refer to epistemological-methodological principles involved in cross-classification and conceptual integration, but each from the perspective of their applications in different domains of science (e.g. logic, mathematics, epistemology, classification theory, chemistry, physics, biology, sociology, political science, cognitive science, AI, etc.). Furthermore, several of these terms – either explicitly or implicitly – refer to *models* and *modelling*. This is understandable, since the very notion of a model – as per the general "definition" that we employ in this study – is based on the coordination of concepts.

* * *

On the basis of the epistemological-methodological concepts outlined in the previous three chapters, we can now trace out four phases of the historical development of conceptual modelling and combinatorial heuristics, as this has evolved through the works of Plato, Ramon Llull, Gottfried Leibniz, and finally Fritz Zwicky, the modern promoter of Llullian-Leibnizian combinatoric modelling – so-called General Morphological Analysis.

4. Plato, cross-division and the genesis of modelling theory

> ”Plato's theory of Forms, the earliest metaphysical theory of classification, is still the paradigm of all typological classifications.” (Stephan Körner[91])

4.1 The first theory of modelling

Plato’s “Theory of Forms” has served as the foundation for three basic branches of philosophy: ontology, epistemology and ethics.[92] From our modelling theoretical standpoint, we are not concerned with the *ontological* status of universals, or the *ethical* issue of universal norms, but with an *epistemological* “theory of concepts”. Like the Theory of Forms in general, its epistemological version allows for the spatial metaphor of concepts as bounded spaces or “containers”. As explored in the previous sections concerning chiastic “squares of opposition” and the “square of classification”, this in turn allows for a diagrammatic representation of certain formal classificatory and cross-classificatory relationships.

The main purpose of this section is to show how Plato applies these basic relationships for generating both Linear Hierarchal Classification (LHC/proper taxonomic structure) and Combinatorial Cross-Classification (CCC/typological structure). Moreover, to the extent that Plato explicitly employs cross-classificatory techniques in order to create complex concepts and relationships, then from the perspective of the minimal definition of a *model* presented in §1, Plato may be regarded as having developed a first (prototype) theory of modelling.

This, of course, does not mean that Plato was the first “modeller”. Modelling – in the most general sense of the term – is ubiquitous to the human condition. Moreover, as concerns “scientific” modelling, Plato in fact credits the Pythagoreans for having handed down the basic notion and art of dividing and combining concepts.[93] Furthermore, there are numerous explicit examples of chiastic modelling pre-dating Plato.[94] However, we can nonetheless credit Plato with being the first (“on-written-record”) to deliberately articulate a theory of cross-classificatory modelling; of presenting case studies as explicit pedagogical examples; and of coining the first general, technical term for a “model”: *paradeigma* (pl. *paradeigmata)* – essentially a *comparative analogous structure*.[95]

Now, the “science” (*episteme*) which is concerned with such classificatory and cross-classificatory relationships – the “communion of kinds” – Plato calls *dialectic* (*dialektike*).[96] And a person who has the competence to conduct such a science is called a *dialectician* (*dialektikos*). Unfortunately, there is much confusion associated with these terms, an issue which needs to be addressed first.

- **The D-word**

The terms “dialectic” and “dialectical” are among the most abused and corrupted concepts in the history of science.[97] Authors have thrown the concepts around indiscriminately, often without even attempting to inform the reader what they think that they mean by them. The

classical scholar J.D.J. Evans expresses this tactfully:

> ”The notion of dialectic is a piece of intellectual currency which ... is more used than understood. Most of those who use it are aware of it only in its more recent cultural forms and are unfamiliar with its historical genesis among the philosophers of ancient Greece.”[98]

Nicolas Rescher is less diplomatic:

> “Modern discussions of dialectics often exhibit a lamentable indifference regarding exactly what is at issue here: what dialectic is; how it works; the procedural methodology of the ways and means of proceeding dialectically – all these are matters of unconcern. ... The ancients, to their credit, do not open themselves to this complaint. Plato provides ample examples of dialectical procedure – quite sufficient to indicate what he sees to be at issue.”[99]

As concerns the epistemological “theory of ideas”, Plato uses the term both in a very general way and in two more specific ways. In its most general sense, *dialectic* is about the analysis and synthesis of concepts and the different possible types of relationships between concepts. This most general expression of *dialectic* has been considered as having almost meta-epistemological status as a “philosophical underpinning of science” as a whole.[100]

Within this general notion, Plato also uses the term in two more specific methodological contexts associated with the two different modes of analysis and synthesis: *propositional* and *compositional*: 1) the *Method of Hypothesis* (laid out in Phaedo and Republic) describes analysis and synthesis as a reciprocal propositional (regressive-progressive) process of abductive-deductive reasoning, e.g. of hypothesising first causes or principles, and then evaluating their consequences; and 2) the *Method of Divisions and Collections* (laid out principally in *Sophist* and *Statesman* and usually referred to simply as *diairesis*) which describes analysis and synthesis as a reciprocal compositional (whole-part) process, i.e. of de-composition and re-composition, and a method for classifying and cross-classifying concepts. Thus in both cases, “dialectic” is characterised by the reciprocal interaction between adjacent conceptual levels (a.k.a. recursive-hierarchal systems, vertical loops and stratified interaction[101]).

It is this second “mode” – *compositional* analysis and synthesis – which is our main point of interest: i.e. the notion of how concepts can be decomposed in order to support definition by classification, and then re-composed and re-combined to facilitate conceptual integration – literally the “intertwining of concepts” (*symploke eidon* – see §4.4).

4.2 *Diairesis* and the three basic logical operators

As was discussed in §2, elementary chiastic structure involves formal relationships which can be expressed by three logical “operators”. Such operators are heavily incorporated in the Platonic dialogues in general, and especially in the four “late” dialogues (*Parmenides*, *Sophist*, *Statesman* & *Philebus*), where they are central to the *diairetic method*: i.e. the “dialectical method of divisions and collections”[102]. Fig. 4.1 shows the three unitary (or “atomic”) logical operators in question:

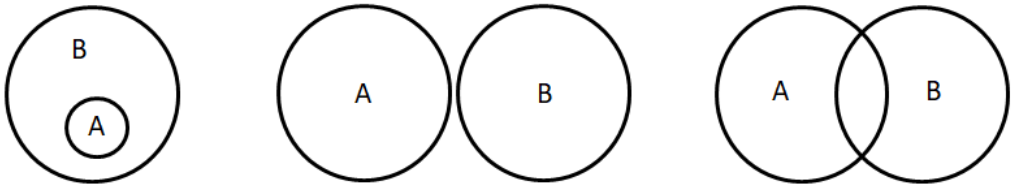


Fig. 4a:
Subordination
All A is B

Fig. 4b:
Disjunction
No A is B

Fig. 4c:
Conjunction
Some A is B
Some A is not B

Figure 4.1: Three "atomic" logical relationships

Now, there has been some disagreement among scholars as to what is, and what is not, to be included under the rubric of the *diairetic method*. Does it 1) consist only of Linear-Hierarchal Classification (LHC) or does it 2) *also* include the boarder notion of the "communion of kinds" as involving Combinatoric Cross-Classification (CCC). That is to say, does it *only* include *sub-ordinate* (Fig. 4.1a) and *disjunctive* (Fig, 4.1b) relationships, or does it *also* include *conjunctive* relationships (Fig. 4.1c)?

As interesting as this question may be for historical reasons, this is not an epistemological issue, but a matter of (nominal) definition. It makes no difference to the subject addressed here. What is important is that we find both of these classificatory structures (LHC and CCC) being employed in the Platonic dialogues, and in the *Sophist* and *Statesman* in particular.

However, there are good grounds for regarding both of these structures as being part of the *diairetic method*, since they both employ the operations of *division* and *collection* (i.e. compositional analysis and synthesis), although their priorities are reversed. LHC is mainly focused on compositional (analytical) division. Although Plato does refer to "collection" in various cases of the diairetic process, these mostly consist of examples of the "things" which make up the form or class *to be divided*. "Collection" seems to consist mainly of identifying a generic concept which can be used as a starting point for hierarchal division. CCC, on the other hand, emphasizes classification on the basis of combinatorial synthesis. Its initial "division" is simply the single-level compositional analysis of two or more concepts (thus creating two or more category variables), which are then cross-ordered (dimensionally coordinated) to produce a matrix of complex concepts.

Although linear-hierarchal classification has tended to dominate the Platonic literature as concerns *diairesis*, the importance of Plato's use of combinatorial cross-classification was already pointed out by Neoplatonic scholars of late antiquity (e.g. Porphyry (234–305), Proclus (412–485); Damascius (c. 462–538) Simplicius (480–560), see below). It has likewise been pointed out by modern classical scholars, among them Benjamin Jowett (1892), Robert Brumbaugh (1952), Kenneth Sayre (1969) and J.M.E. Moravcsik (1992).[103]

More recently, a number of scholars have begun to consider Platonic *diairesis* as consisting of a whole variety or range of methods.[104] For instance, Ionescu (2013) and Smith (2020) regard

the famous "dialectician passage" at Sophist 253d1-e3 as involving both hierarchal division and conjunctive "communion of kinds"[105] (see §4.3, *infra*). Pavani (2023) bases different variations of *diairesis* on their functions or goals – e.g. whether it is being used to define a concept ("definition by classification"), to construct a full taxonomy, or to explore the possibilities of combining concepts in order to generate new sets of compound concepts. And Liu (2021, 2024) has explored in detail Plato's use of both dichotomous and polychotomous cross-divisions as chiastic modelling (see §4.4, *infra*). These "variations" of *diairesis* are summarized in Fig. 4.2.

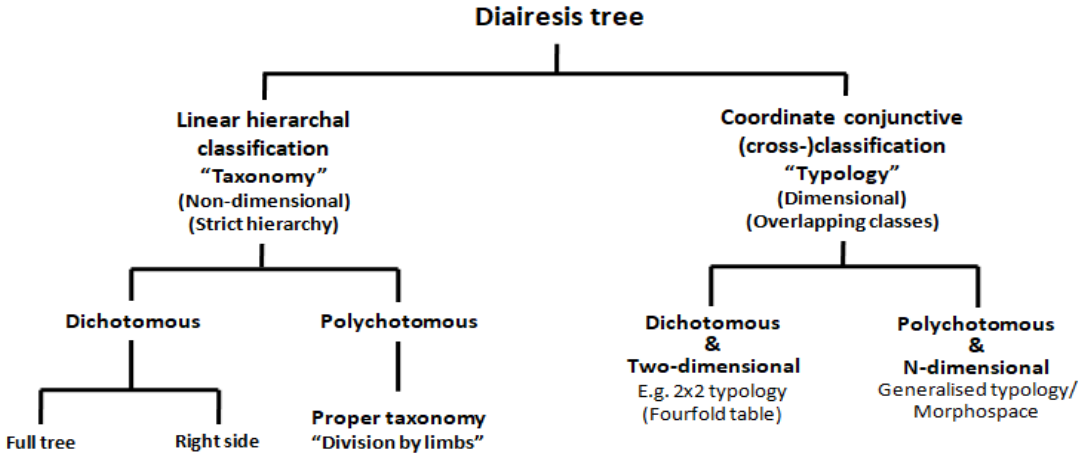


Fig. 4.2: Diairesis tree

However, simply *employing* such relationships is not the same as proposing an outright theory of their use. Firstly, it would be nice to find a concise passage in which Plato describes all three of the logical relationships (Fig. 4.1) together – as a complete "operational toolbox" so to speak. Secondly, we also need to find text where he explicitly – and *didactically* – presents combinatoric "models" in the form of cross-classificatory matrices. Fortunately, we have a good candidate for the former (§4.3) and explicit examples of the latter (§4.4). We begin with the "operational toolbox".

4.3 The "dialectician passage" in Sophist 253d-e

The dialogue *Sophist* is a pivotal work in ancient philosophy of science. It includes some of the most inspiring – and enigmatic – passages in the Platonic dialogues and has fascinated classical scholars and philosophers for centuries.[106] The so-called "dialectician passage"[107] (*Soph.* 253b-e) is one of the most famous of these. It has been an object of debate for centuries and is still a hot topic. It seems to describe – in terms of Platonic forms – four different ways that "ideas" (or concepts) can and cannot combine. Now, modern scholars have pointed out that this passage obviously concerns more than *merely* describing "four different kinds of relations between Ideas".[108] It is seen as being part of a more complex and sophisticated account of ontological relationships taken up later in the dialogue involving the so-called "five great kinds".[109]

I totally agree with this and fully appreciate the fact that Plato had bigger fish to fry than simply describing "four different kinds of relations between Ideas". However, to state that the dialectician passage does not "merely" concern itself with describing such relationships does not preclude the possibility that the passage nonetheless *does – also –* describe them. For, regardless of what fish you are frying, you will be in need of certain basic utensils for the job and in the case of the "communion of kinds" (or the "co-mingling of concepts"), these utensils include the use of the three basic "logical (relational) operators" (Fig. 4.1, *supra*). And such operators are clearly present in the conceptual structures that Plato is describing in this passage, whatever its wider purpose.

The dialectician passage starts off at 253b-c by stating that, if we agree that "kinds" (forms, classes or concepts) can "co-mingle" or "join together" (συμμείγνυσθαι), then the dialectician has to have the expertise to discern between such "kinds", and how they can and cannot combine.[110] Then comes the famous sub-passage at 253d5-e2:[111]

> "*Visitor*: Then he who is able to do this has a clear perception
>
> **(R1)** of one form extending entirely through many individuals each of which lies apart, and
>
> **(R2)** of many forms differing from one another but included in one greater form,
>
> … and again [αυ]
>
> **(R3)** of one form evolved by the union of many wholes, and
>
> **(R4)** of many forms entirely apart and separate.
>
> This is the knowledge and ability to distinguish by classes how individual things can or cannot be associated with one another."

This passage has been examined, interpreted, bandied about – and in some cases simply given up in bewilderment[112] – in scores of studies during the past century. It has been subject to many various translations and, at the same time, to different interpretations, even within one and the same translation. My aim here is neither to evaluate translations nor their interpretations – which I am certainly not qualified to do – but to look at the possible classificatory relationships being treated in these statements from a purely modelling theoretical perspective.

The four relational structures are usually partitioned into two groups R1+R2 & R3+R4, separated by the seemingly significant "αυ" ("and again"):

- **R1+R2: Subordinate relationships**

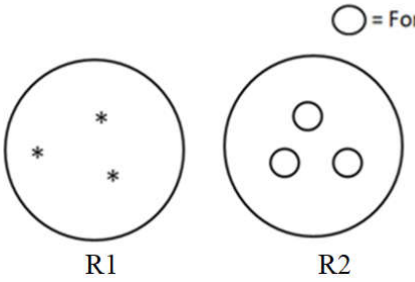


Figure 4.3: Particulars vs. forms

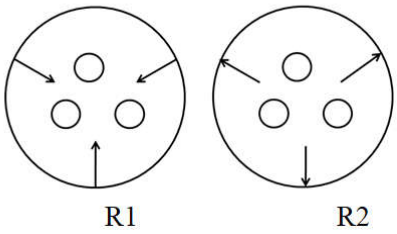


Figure 4.4: Divisions vs. Collections

R1 and R2 have been interpreted in two different ways (Figs. 4.3 and 4.4 respectively) as describing subordinate-superordinate relationships:

I. (**Figure 4.3: Particulars vs. Forms**): The subordinate entities contained in R1 are *particular instances* of the superior form; whereas in R2 the subordinate entities are themselves (sub-) *forms*. This is justified on grammatical grounds, i.e. that the gender of the terms used to denote the subordinated entities in R1 points to them being particular "things".

II. (**Figure 4.4: Forms only**): Both R1 and R2 concern *forms*. R1 proceeds from a superior form to the subordinate forms, indicating (compositional-analytic) "division"; whereas R2 proceeds from subordinate forms to a superordinate form, indicating (compositional-synthetic) "collection". This is justified on the grounds that it is generally accepted, that Plato is only concerned with "forms" in *this account* in the Sophist, not "particulars".

I prefer the second alternative but, for present purposes, either explanation will suffice.

- **R3+R4: Co-ordinate relationships**

We take R4 first, which is relatively straight forward, before tackling the more cryptic R3. R4 seems to represent a group/set of free-range *disjunct concepts* (or genera) within a particular **U**niverse of **D**iscourse [**U**] (Fig 4.5). If this set is exhaustive, it can be rendered as a *partition* of that universe (Fig. 4.6). In any event, it corresponds to Fig. 4.1b (*supra*), i.e. *disjunct coordinate concepts*. There is no sub-ordination or hierarchy involved here, nor any overlapping (conjunction) of concepts.

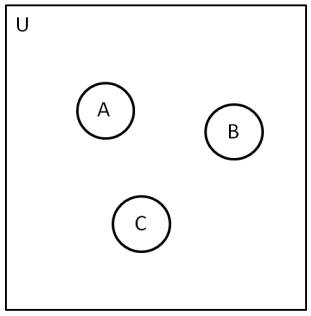


Figure 4.5 Three disjunct genera

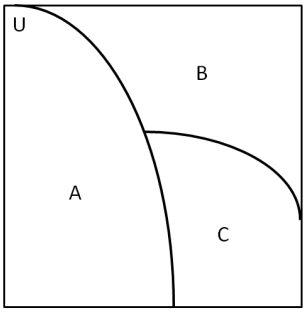


Figure 4.6: 3-Partitioned **U**.

Finally, the case of **R3**. Let us cite it again in Fowler's translation.

(R3**)** …one form evolved by the union of many wholes… (Fowler (1921))

The Greek text is: *kaì mían aũ di' hólōn pollõn en henì sunēmménēn*. The word *sunēmménēn* is generally translated as "join together", "compose" or "combine". To get a better perspective on this we can look a number of other translations:

- (R3) "a Form passing through several 'wholes'… combined to form a 'unity'..." (Gomez-Lobo, 1977).
- (R3) "one unified [Form] composed of many wholes…" (Smith, 2019).
- (R3) "a single form that's connected as a unit throughout many wholes..." (White, 1997).
- (R3) "one [Form] through many wholes gathered into a one..." (Smith (2020))
- (R3) "a single [form] [running] through many wholes that is gathered into a one …" (Miller, 2016).
- (R3) "a single Idea linked in a unity extending over many whole Ideas…" (Trevaskis, 1967).
- (R3) "one form *knit together* into a single whole and pervading many such wholes…" (Jowett, 1892).

All of these, taken together, reek of "conjunction". Indeed, many classical scholars – especially in connection with the Sophist and Statesman – frequently refer to cross-division and "overlapping concepts", although usually they do not go into much further detail (see §4.4, *infra*).

Consider Figure 4.7 – a rather mundane but nonetheless functional example of "overlapping concepts" or "co-ordinate conjunction". The three properties (A: mammal; B: hunts fish; C: can fly) are fully co-ordinate, i.e. they generate a full 3-dimensional combinatorial lattice or "blend". This is a simple three-(binary)-variable cross-classification creating $2^3 = 8$ species.

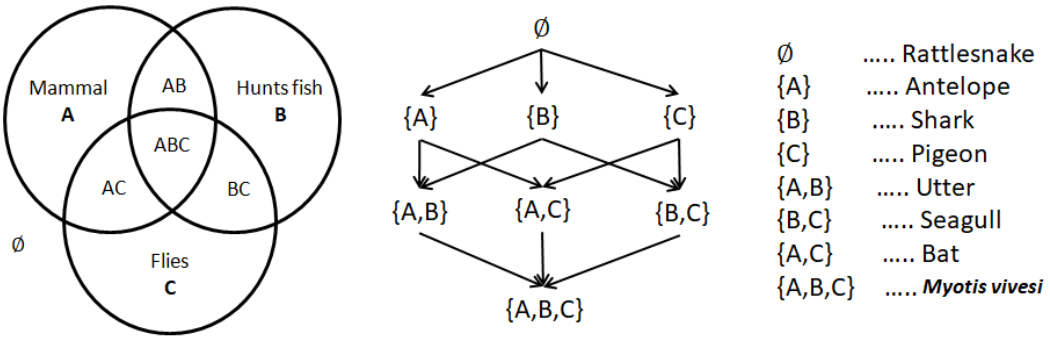


Figure 4.7: Three-property conjunction with its meet-lattice and example list.

If the four relationships in the dialectician passage can be interpreted as being based on the three logical operations in Fig. 4.1 (R1 & R2 being two ways to approach *subordination*), then we can present the passage as in Figure 4.8.

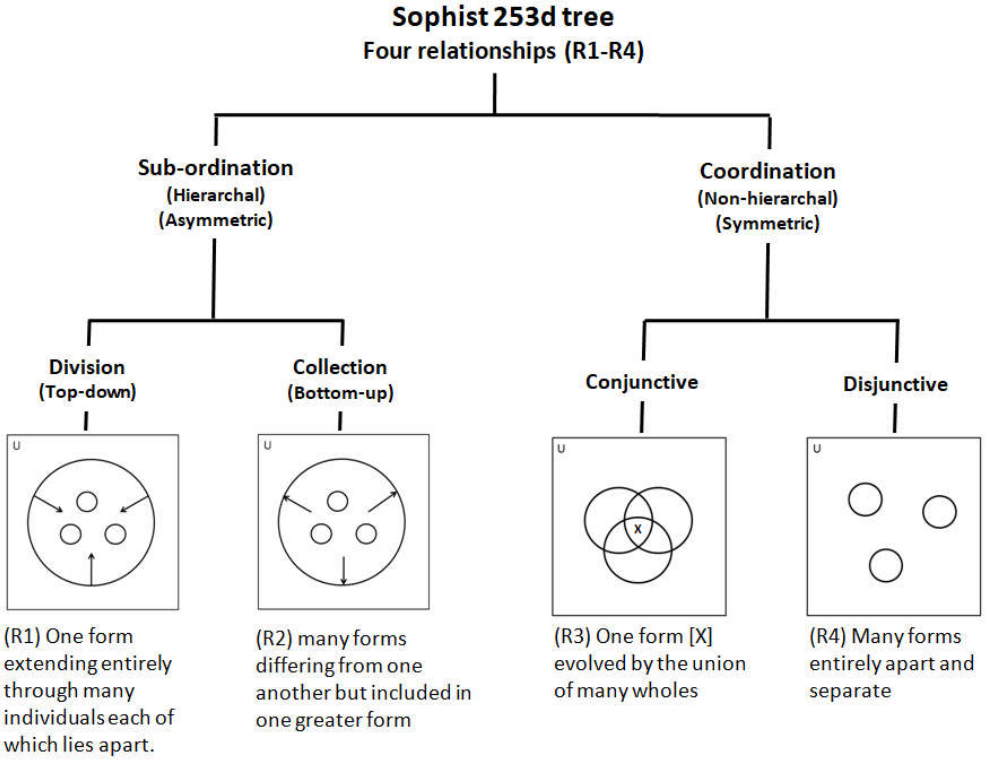


Fig. 4.8: Dialectician-Passage Taxonomy

Summa Summarum: The four relationships in the "dialectician passage" have been classified here according to a taxonomy split along two main branches: (1) asymmetric linear hierarchal *sub-ordination* vs. (2) symmetric non-hierarchal *co-ordination*. (1) is further divided into top-down *division* (R1) vs. bottom-up *collection* (R2); and (2) is further divided into *conjunctive* (R3) vs. *disjunctive* (R4) co-ordination. To what extent this is a possible (alt. plausible) interpretation of the "dialectician passage", I respectfully leave to the classical scholars.

However, I do not believe that this interpretation is actually anything essentially new – not even in the case of **R3** as expressing *intentional conjunction*. Indeed, this seems to be alluded to by several classical scholars. Furthermore, since Plato explicitly, and in a clear pedagogic manner, constructs cross-classificatory "typologies" elsewhere – in both *Sophist* and *Statesman* – then we know that he was already familiar with this type of chiastic modelling, to which we now turn.

4.4 Cross-classification and typological modelling in the late dialogues

> "… Plato's divisions are quite unlike the Aristotelian classifications. For not only is there the difference in terms of need for instantiation, there is also the difference of Aristotle needing to carve out non-overlapping species and hierarchies of genera, whereas in Plato's scheme the intermediate cuts can *crisscross* the generic Form of Art in a variety of ways."[113]

Throughout the dialogues, Plato uses several different terms which have been translated variously as mixing, blending, combining, joining together, connecting, communing, sharing, interrelating and co-mingling.[114] One of the most expressive of these terms is *symploke eidon*, the intertwining or combining of ideas or concepts. (Note that the Latin *calque* of the Greek "sym-ploke" is "com-plex".) Sayre (1969) – a classical scholar also schooled in mathematics – confirms that "... combination among the Forms as depicted in the Sophist is a *symmetrical* relationship"[115], not an asymmetric-subordinate one. Van Fraassen (1969) – not primarily a classical scholar but a philosopher of science, epistemologist and logician – has likewise noted that such concepts must represent logical conjunction, corresponding to "set intersection".[116] And, as discussed earlier in §3, any (extensional) intersection of sets or (intensional) conjunction of concepts automatically produces a chiastic (typological) structure:

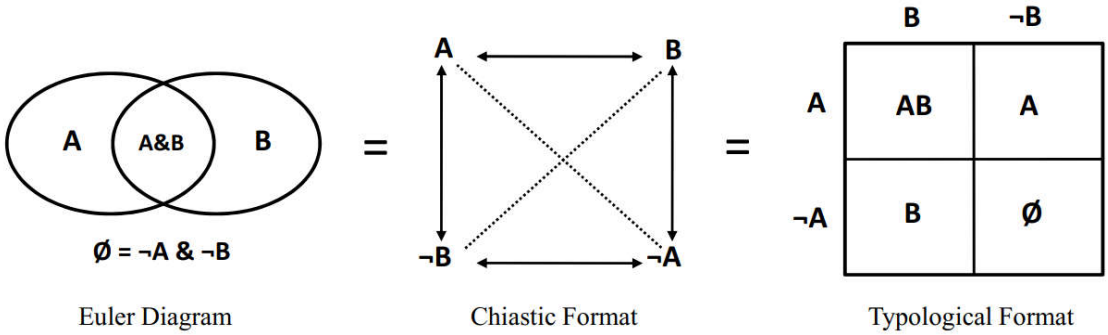


Figure 4.9: Three formats for binary concept conjunction

4.4.1 The four-fold typology in Sophist

Some 12 Stephanus pages after the "dialectician passage", near the end of the dialogue at *Soph*. 265–266, we find the Eleatic Visitor giving his interlocutor Theaetetus a lesson in cross-classification. The visitor starts out by making a distinction between two bisociate pairs of contrary concepts concerning the "production of things": On the one hand, between that which is produced by nature itself (*divine*) vs. that which is produced by humans (*artefacts*); and, on the other hand, between *original*s, i.e. the actual "things-in-themselves" (*noumenon*) vs. how these things appear to the senses as *copies* (*phenomena*). This can be represented as parallel division (a.k.a. quasi-tree – Figure 4.10).

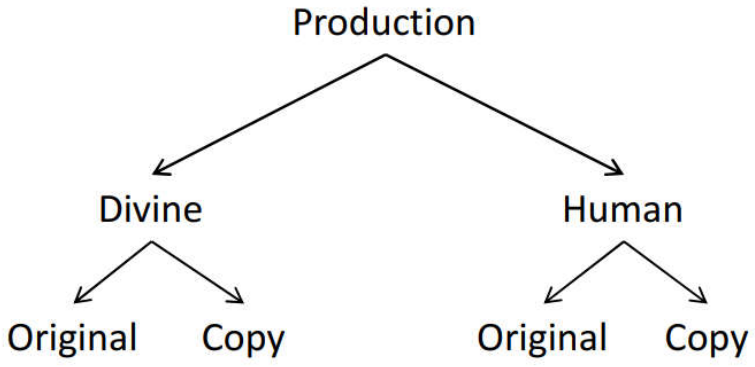


Fig. 4.10: Parallel division of Divine and Human production

Here is the text, starting at *Soph*. 265e3, where Plato turns this "quasi-tree" into a cross-classificatory (typological) matrix[117] :

> "I will assume that *divine expertise* produces the things that come about by so-called nature, and that *human expertise* produces the things that humans compound those things into. According to this account there are two kinds of production, *human* and *divine*."
>
> "Now ... first divide the two kinds of production width-wise [horizontally], so to speak [Figure 4.11a]; now divide it [again] lengthwise [vertically] [Figure 4.11b]. In this way we now get four parts in all; two belong to us and are *human*, and two belong to the gods and are *divine*."
>
> "And again, when the division is made the other [vertical] way, one part of each half has to do with the making of real things [*originals*], and the two remaining parts may very well be called the making of images [*copies*]; and so productive art is again divided into two [other] parts." [Figure 4.11c]

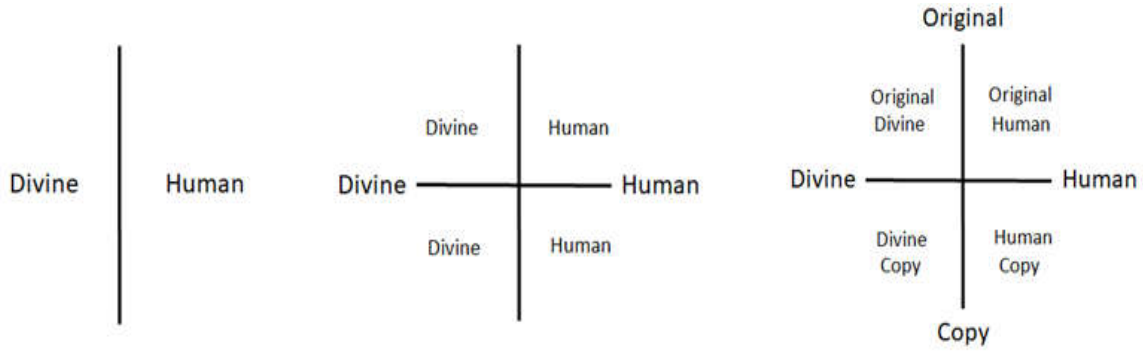


Fig. 4.11a
Horizontal divided
Vertical cut

Fig. 4.11b
Vertical divided
Horizontal cut

Fig. 4.11c
Four-fold cut
Chiastic cross-cut

Figure 4.11: Production of the four-fold classification at Sophist 265e3.

When the penny finally drops for Theaetetus, he states:

> "I understand better now; and I agree that there are two kinds of production, each of them twofold – the *divine* and the *human* by one method of bisection, and by the other *real things* and the product that consists of a sort of *likenesses*." (266d5-8)

Plato's dialogues are full of such "verbal matrices" (cf. Brumbaugh, 1954), but this – as far as I know – is the first explicit, didactic presentation of a "four-fold classification", the simplest form of a typological model. It is not known whether Plato actually included diagrams in his original works, but verbal diagrammatic imagery is certainly evident here.[118]

4.4.2 The 2 x 3 typology in Statesman

In the Statesman, Plato has the "Visitor" defining possible political constitutions on the basis of two co-ordinate variables.[119] The first variable concerns *how many* are involved in ruling:

> (291d1) "Visitor: We recognize *monarchy*, don't we, as one of the varieties of rule in cities? … After monarchy one would, I think, list the holding of power by *the few*. … And isn't a third type of constitution rule by the mass of the people, called by the name of *democracy*?

The second variable concerns whether rule is carried out on an equitable, just and lawful basis – in short, whether it is governed *legally* or *illegally*:

> (291e1) "I think that as things are, people refer to the aspects of force and consent, poverty and wealth, and law and lawlessness … and use these to divide each of the first two types into two. So they call *monarchy* by two names, on the grounds that it exhibits two forms, the one '*tyrannical*', the other '*kingly*' monarchy. … And any city which has come to be controlled by a few people they call by the names of '*aristocracy*' and '*oligarchy*'.

Then, at 302c-e, the Visitor puts it all together:

> "…there are three sorts of constitution, monarchy [rule by one]... rule by a few and rule by many … Let us, then, by dividing each of these into two parts, make six. … We said that monarchy comprised *royalty* and *tyranny*, and the rule of the few comprised *aristocracy* … and *oligarchy*; but to the rule of the many we gave then only a single name, democracy; now, however, that also must be divided ... [by] ruling according to laws and contrary to laws."

Rulers --> Legality	One	Few	Many
Legal	kingdom	aristocracy	democracy
Illegal	tyranny	oligarchy	democracy

Figure 4.12: 2 x 3 typological matrix of political constitutions[120]

4.4.3 *Parmenides* and the 3-dimenasional typology

As discussed in §3, typologies can be extended in two ways: (1) by extending the *number of elements* in one or more of its dimensions (or variables); and/or (2) by increasing the *number of dimensions*. We see an example of the former in the two cases presented above showing a 2x2 and a 2x3 typology. Now we can present an example where Plato – in the dialogue *Parmenides* – constructs a 3-D (2x2x2) typology. Earlier researched by Sayre (1978, 1983), with additional contributions by Mienwald (1991, 2014) and Gill (2014), it has recently been further elaborated by Liu (2024), who shows how the set of eight hypotheses is produced by chiastically cross-ordering three types of relationships between "the One" and "the Many".[121]

Many regard the *Parmenides* as being the most enigmatic of all the Platonic dialogues, especially its second section in which Plato presents what he calls a "school exercise" in the dialectical *communion of forms*. It is a challenging "exercise" and there is some divergence concerning just how to regard it. Some consider it as a kind of satire, where Plato is making fun of an overly-pedantic combinatory play with seemingly simple (but in fact seriously confusing) universal concepts. However, most regard it as a genuine exploration into of the logical relationships between such concepts and, indeed, a (self-) critical analysis of Plato's own "theory of forms". Personally I prefer the latter, although these two motives need not be mutually exclusive. However, from our "modelling theoretical" perspective it makes no difference: what is important is that this is a deliberate exercise in multivariate conceptual modelling, regardless of what one may think of the subject matter itself. Here is the passage in which Plato proposes his (gymnastic) "school exercise":

> "... whatever you might hypothesize as being, or as not being, *or as having any other property*, you must examine the consequences for the thing you hypothesize in relation to itself and in relation to each one of the others, whichever you select, and in relation to several of them and to all of them in the same way; and, in turn, you must examine the others, both in relation to themselves and in relation to whatever other thing you select on each occasion, whether what you hypothesize you hypothesize as being or as not being. All this you must do if, after completing your training, you are to achieve a full view of the truth." [To which Socrates immediately rejoins with "Scarcely manageable, Parmenides, this task you describe!".][122]

On the basis of this passage, three bisociate pairs of complementary concepts (two *antecedent hypotheses*, two *subjects* and two *relationships*), labelled A, B & C, are conjoined (Fig. 4.13). The *typology* of the eight hypotheses is shown in Fig. 4.14.

{**A**} if the One is [vs.] {¬**A**} if the One is not (Antecedent hypotheses)
{**B**} the One [vs.] {¬**B**} the Others (Subjects)
{**C**} in relation to the One [vs.] {¬**C**} in relation to the Others (Relationships)

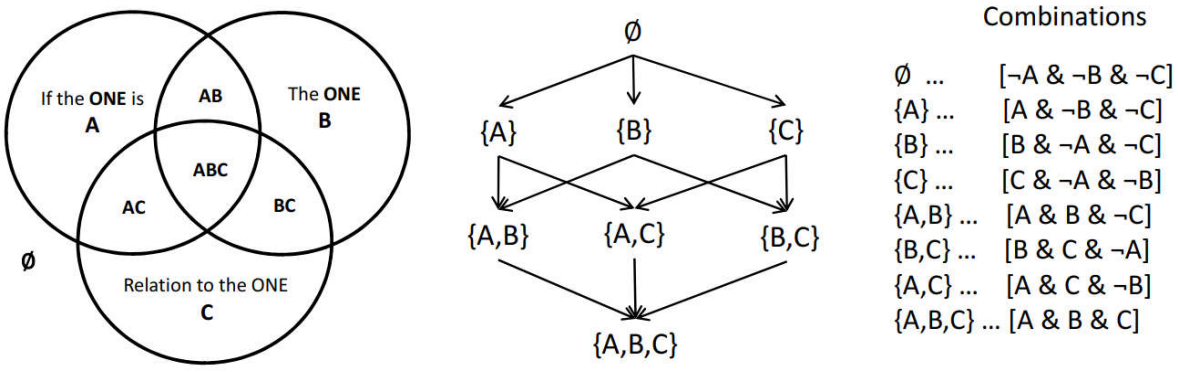


Figure 4.13: Euler diagram & meet-lattice with combinations

	If the ONE IS A		If the ONE is NOT ¬A	
	In Relation to the ONE C	In Relation to the OTHERS ¬C	In Relation to the ONE C	In Relation to the OTHERS ¬C
The ONE B	I	II	VI	V
The OTHERS ¬B	III	IV	VII	VIII

Figure 4.14: {2x2x2} embedded typological matrix producing 8 hypotheses.[123]

4.4.4 The misconstrued four-fold classification in Philebus

Our final example of Platonic cross-classicisation is taken from Philebus, but here we find it to be the victim of a misunderstanding in the literature. We treat this case because it helps to illuminate not only the issue of cross-classification, but also that of cross-disciplinary understanding and communication.

The problem taken up in the Philebus concerns the relationship between two central "properties" of the human condition: *knowledge* (i.e. "real" knowledge or *episteme*, not mere opinion) and *pleasure*. The question is, which of these two is most indicative of the "good life"? The interlocutors (Socrates and Protarchus) agree that if one is forced to choose between the two, then it is ("real") *knowledge* which is superior. However, they also agree that these two properties are not mutually exclusive and can be combined. In that case, there would seem to be three alternatives: 1) there is the possibility of pleasure without knowledge; 2) of knowledge without pleasure; and finally 3) pleasurable knowledge (or knowledge which engenders pleasure). Then Socrates goes on to bring up a forth factor: the *cause of* or *reason for* the blend between the two. It is on the basis of this "fourth factor" that some classical texts call this a "four-fold classification". However, this is misconstrued chiastic logic.

The knowledge-pleasure blend is not a *four-fold* classification because of the "*fourth* factor", i.e. the "cause of the blend". This "factor" is totally extraneous to the four-fold structure itself. As has been noted earlier, when two bisociate concepts or properties are conjoined, a 2x2 typology (a.k.a. four-fold table) is automatically formed (Fig. 4.15).

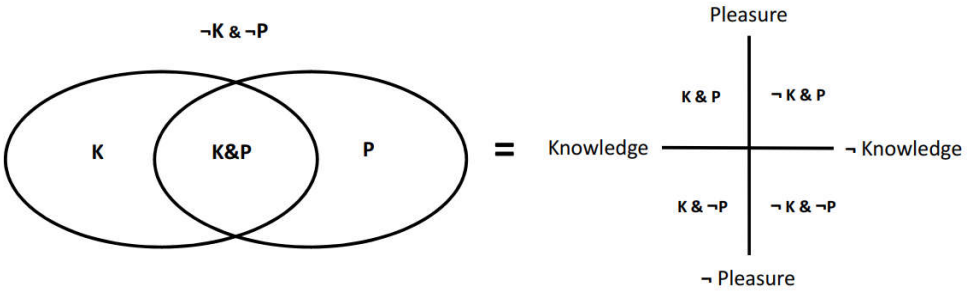


Fig. 4.15: Knowledge-Pleasure cross-classification in Euler and typological format

Thus the "four-fold" structure is already accomplished in the conjunction of *knowledge* and *pleasure*. The so-called "forth factor" i.e. the *reason* for the blend – as interesting and relevant as it may be – is not part of this "fold". The whole idea of a "fold" concerns geometric progression by combinatorial cross-ordering.

I conjecture that Plato doesn't mention the implicit fourth conjunct (¬K & ¬P) because he is, in this particular dialogue, more interested in the substance of the knowledge vs. pleasure issue itself, rather than in the formal aspects of cross-classification *per se*. (Note also that he ponders adding a "fifth factor", although there is no such thing as a "five-fold classification". This supports the view that, while he is promoting conceptual conjunction or "blending", he is not specifically teaching cross-classification *per se* – as in the Sophist.)

Now, this is not an issue of great consequence and I report it simply "for the record". However, it does emphasize the fact that, just as contemporary social scientists may not always be aware of earlier historical versions of certain modern methodological concepts, modern classical scholars may not always be aware of the existence of contemporary concepts that *formalise* and *generalise* ancient prototypes of such concepts. Indeed, the identification of such historically comparable concepts, and how these have developed in the long-term (centuries; millennia) to express analogous or generalised methodological concepts, is one of the central features of *Epistemological History*.

4.5 Porphyry, Simplicius and chiastic "logic"

It is not certain who first started using the term *chiasmus* and "chiastic order" to denote combinatoric cross-classificatory structure.[124] However, some five centuries after Plato's formulation of such structure, the Neoplatonic scholar Porphyry (A.D. 234–305), in his *Isagoge* (an introductory text on classical logic) calls the simplest form of cross-division "chiastic". First (at §71.5), he states that some things "are said *in combination* by being joined by a *coordinating conjunction*". Then, as an example (starting at §78.22), the inquisitor takes up two pairs of contrary concepts, the *universal* vs. the *particular*, and *substance* and *accident*, and asks his teacher how these should be ordered. "Chiastic order" says the teacher...

> "In chiastic order ... to divide in the way, saying that among beings, some are *either* universal substances *or* particular accidents, and some *either* universal accidents *or* particular substances, is to use a chiastic ordering: neither universals or particulars, or substances or accidents, are listed in an order of precedence." [In this way it] "... sets out the combined explanatory accounts belonging to their names". (79, 1–11.)

Simplicius (c. 480–c. 560) – in On Aristotle's "Categories" – approaches the matter in a way that better emphasizes the chiastic *bisociate contrary* square. In explaining the same substance-accident vs. universal-particular coordination as Porphyry, he states: "When these are *woven together*, there comes about a total of six combinations *(sumplokai).* Among these, two are non-existent: *viz*. those which bring together contradictories [i.e. lines 1 and 2]".[125]

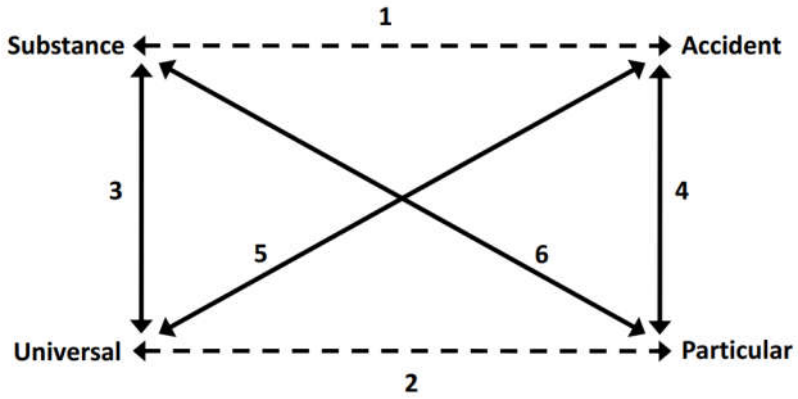


Fig. 4.16: Simplicius' *bisociate contrary* chiastic square.

The translator (classical scholar Michael Chase) notes:

> "These six combinations are arrived at by *chiastic logic* ... which was popular at least since the time of Porphyry. We start with an enumeration of four terms, in this case substance, accident, universal, and particular. They are then arranged in a square, with each term placed at one of its angles. ... The lines with arrows indicate the possible relationships between these four terms (bearing in mind that in Greek, unlike in English, word order is relatively unimportant). We thus end up with a total of six possible combinations *(sumplokai):* (1) accidental substance [contradictory]; (2) universal particulars [contradictory]; (3) universal substance; (4) particular accident; (5) universal accident; and (6) particular substance. Simplicius will now set out to investigate each of these six possibilities of existence." (N. 453, p. 126f. Brackets added.) .... "Thus ... combinations (3), (4), (5), and (6) – which Ammonius *(In De Interp*, 25,8–9) calls the 'subordinate' and 'diagonal' couplings respectively – are considered real logical possibilities." [N. 457]

* * *

So much for ancient chiastic modelling. In the "Western" world, it will take some seven centuries before the next major step in the evolution of cross-order combinatoric modelling takes place. And when it does take place, it is with the help of Hindu and Islamic science and philosophy.

5. Ramon Llull and the combinatorial art

> "The understanding longs and strives for a universal science of all sciences, with universal principles in which the principles of the other, more special sciences would be implicit and contained as is the particular in the universal" (Ramon Lull – *Ars magna et ultima*.[126])

5.1 The mission

The next significant step in the evolution of combinatorial heuristic modelling was taken by the Catalan philosopher-poet Ramon Llull (1232–1316) who would extend Platonic cross-classification by developing what is today called a *morphospace*, i.e. a set of category variables ("tuples") coordinated to produce a multivariate configuration space.

Llull is undoubtedly one of the most enigmatic figures in the history of science and philosophy. Both visionary and quixotic, he succeeded in acquiring two monikers: one the celebrated *doctor illuminates*; the other *vir phantasticus* (crazy man). He was born and brought up under extremely "blended" circumstances at the geographical intersection of two major (hostile) cultures and religions: Christian and Muslim (with the added mixture of Jewish). In addition to this, he lived during a historical-philosophical watershed in which "…no phase of Scholasticism ... remains more confused than the events in the decades immediately each side of the year 1300."[127]

Llull had an aristocratic upbringing, becoming tutor to King James I's son and later steward of the royal household. However, around the age of 30, after some years of aristocratic overindulgence, he had an epiphany: he would dedicate his life to Christianity and, in particular, to the conversion of the Muslims (and Jews) to the Christian faith. However, he was decidedly different from most of the other "converters" of the time: he was totally against the use of coercion and declared that any effort to convert Muslims and Jews to Christianity would remain fruitless if Christian missionaries could not give them a good (i.e. *rational*) reason to do so.

In order to carry out this enterprise, he would need to be able 1) to speak and write Arabic and understand Islamic culture and theology, and 2) have at his disposal a pedagogical method that was not based on "authoritative Christian texts" (scripture) and purely dogmatic beliefs, but on rational and *theologically neutral* arguments. He needed a program for the art of interreligious dialogue, where "everyone could use the same thinking alphabet and the same speaking alphabet".[128] His aim was:

> "… to find a mechanism to prove and generate truths in such a way that, once everyone agreed on the [initial] assumptions, the objectivity of the procedure would force all to accept the conclusions."[129]

To this end, Llull "acquired" a literate Muslim slave and spent nine years learning the Arabic language and culture in preparation for his mission. During this period he studied Arabic

science and literature, including the Neoplatonic texts of Al-Farabi, Ibn Sina (Avicenna) and Al-Tusi.[130] This background in Islamic philosophy and science was crucial for Llull's achievements. He adopted and adapted themes from Islamic mathematics and logic first into theology, and then proposed their generalisation into a heuristic modelling procedure for *science as a whole*.

He also studied ancient Greek philosophy including (importantly) the Neoplatonic texts of John Scotus Erigena and Pseudo-Dionysus, the latter a Christian interpreter of Proclus, the most influential of the ancient Neoplatonists for early modern science.[131] This "private education" resulted in Llull being forever an academic outsider, but at the same time it left him academically unfettered to devise his own methods.

In order to be able to argue rationally about the articles of Christian faith, Llull needed theologically neutral premises – concepts which could be agreed upon by all three faiths – from which he then could carry out rational demonstrations. He needed to re-axiomatize Western monotheist theology by finding basic premises from which valid logical demonstrations could be inferred, thus *combining* the reciprocal methods of *ars inveniendi* and *ars demonstrandi*.

> "If now we try to analyze the techniques Llull is using here, we will see first of all that he is not ... working according to the Euclidean model, where a group of *pre-established* principles are used to prove successive new principles (theorems) [the geometric or axiomatic-synthetic method]. Instead he presents the question as a *hypothesis*, and draws out the implications of assuming this hypothesis to be true or false. ... Rather than using principles to work *towards the thing to be proved* [synthetically], he is, as it were, *working backwards*, starting with the hypothesis to test it against the principles [analytically]."[132]

This is Plato's hypothetical-analytic (and Peirce's abductive or retroductive) method, i.e. so-called "*regressive analysis"*, the working back from conclusions to first principles, premises or causes by means of which something can then be derived or explained (by the reciprocal process of 'synthesis')".[133] But then Llull does something new: he *explicitly* combines this with *compositional* analysis and synthesis to further develop combinatorial, cross-classificatory techniques:

> "... each complex concept that we conceive is analyzable into all its component parts, down to the most simple ones. Simple, i.e. un-analyzable, concepts are the 'primitives' from which all compound concepts originate. Once one arrives at the primitives, one may choose an alphabet and attribute a sign to each primitive: thus, by combining signs, one will obtain all the complex concepts in a purely mechanical way. Analysis, synthesis, and the discovery of an appropriate alphabet were the main stages of this program aiming to renew logic as both an art of judgement (the ars iudicandi [demonstrandi]) and an art of discovery (ars inveniendi).[134]

5.2 Islamic influence on Llull's combinatorial art

Although there has been some debate on the matter of just how much the Greeks concerned themselves with combinatorial studies,[135] the fact remains that relatively little work in this

area *per se* was passed down directly from ancient Greek and Latin sources to medieval and Renaissance Europe. Indeed, one of the latest anthologies on the subject – "Combinatorics: Ancient and Modern"[136] – contains introductory sections on *Ancient Combinatorics* which take up Hindu, Chinese, Islamic and Jewish contributions, with the conspicuous absence of any significant Greek or Latin contributions.[137]

The earliest examples of combinatorial mathematics are usually ascribed to the ancient Indian mathematician Pingala in the *Chandaḥśāstra* (or *Pingala Sutrasc*) from c. 200 BCE. What drove this development was "a phonologically structured syllabary such as that of Sanskrit for a mathematical study of prosody, the science of *chandas*".[138] That is, the study of permutations and combinatorial methods of enumerating poetic and musical metre in a syllabic script (see below). Indeed, the letters of a phonetic script – whether syllabic or alphabetic – quite naturally lend themselves to combinatoric analysis (as with Plato and his epigones somewhat earlier).

During its 7th century expansion, Islamic culture came into contact with Hindu arts and science, which eventually allowed Islamic mathematicians to combine Hindu and classical Greek mathematics. Thus, in eastern Arabic territories we find the expression for "arithmetic" being called *ḥisāb al-hindī* (Hindu reckoning).[139]

During the Islamic Golden Age (c. 750–1300), its philosophers and mathematicians further developed all of the constituents that would show up in Llull's Great Art. This includes:

- the beginnings of a (mathematical) *theory of combinations and permutations*;
- the use of an *abbreviative letter symbolism*; and
- the appreciation of the reciprocal nature of discovery (*ars inveniendi*) and proof (*ars demonstrandi*), including their connections with *analysis* and *synthesis* [140]

Finally, there is the *Zairja*. As early as the 10th century the Islamic philologist and grammarian Ibn Durayd (837–933) wrote of a method of combining letters in different ways by arranging them in concentric rings and then turning the rings to produce different combinations. Llull knew of this "divinatory device" called a *Zairja* (*zā'irjah*) as attributed to a Maghribi Sufi at the end of the twelfth century.[141] It was constructed from paper and could be manipulated to process letter symbols in order to produce many different word combinations from a relatively small initial set of letters. Ibn Khaldūn described this as "a branch of the science of letter magic … the technique of finding out answers from questions by means of connections existing between the letters of the expressions used in the question."[142] Such "question-answering disks" were known to be on sale in marketplaces in Algeria in the 1260's and 70's.[143]

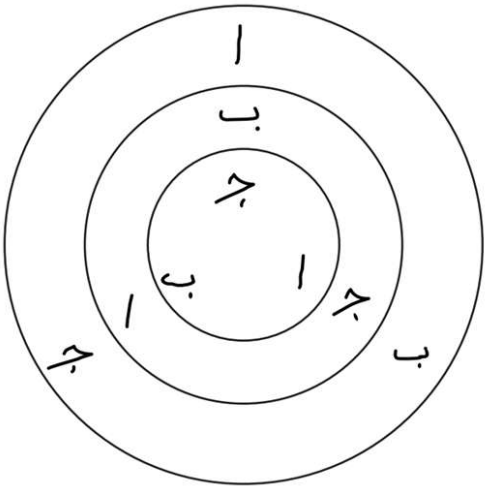


Figure 5.1: Rotating rings of Ibn Durayd[144]

In the 16th and 17th century, Llull's examples of movable *Zairja* had become "an intellectual obsession in European culture".[145] They were known as *Volvelle* – literally "revolving vellum" – and were incorporated in so-called "movable books" as mechanical learning devices.

This background in Islamic influence is not meant to diminish Llull's achievements. He adopted and adapted these themes from Islamic mathematics and logic first for theological purposes, and then attempted – or at least proposed – their generalisation into a heuristic modelling procedure for science as a whole. Furthermore, as concerns the *Zairja*, Llull was not interested in combinatoric letter or word *poiesis*; he was interested in combinatoric *proposition poiesis*. He wanted to generate new propositions that could then be ordered and employed as premises for deductive inference.[146] He wanted to produce an *inveniendi-demonstrandi* "logic-machine". Indeed, in face of the historical evidence of his influence on Renaissance and Enlightenment science, Francis Yates went so far as to say: "The European search for method . . . began with Ramon Lull."[147]

5.3 The Great Art

In order to be able to argue rationally about the articles of Christian faith, Llull needed theologically neutral premises – concepts which could be agreed upon by all three faiths – from which he then could carry out rational demonstrations. Thus his point of departure for his Great Art was not Christian scripture, but a hypothesis about – and an analysis of – the most basic theological principles which are common to all three Western monotheistic religions. At the head of these first principles were the "primary attributes" or "dignities" of God which are found in the Christian works of Pseudo-Dionysius[148], in the Islamic *Hadras* and the Judaic *Sefirot*, all of which were heavily influenced by Neoplatonic tradition.[149]

> "The religious principles upon which Lull based his Art, which were held by all three religious traditions, was the importance which Christian, Moslem, and Jew attached to the Divine Names or Attributes. The Attributes, or, as Lull prefers to call them, the *Dignities of God* on

> which the Art is based are Bonitos (Goodness), Magnitudo (Greatness), Eternitas (Eternity), Potestas (Power), Sapientia (Wisdom), Voluntas (Will), Virtus (Virtue or strength), Veritas (Truth), Gloria (Glory). Religious Moslems, Jews and Christians would all agree that God is good, great, eternal, powerful, wise, and so on. These Divine Dignities or Names, combined with elemental theory, gave Lull what he believed to be a universal religious and scientific basis for an Art so infallible that it could work on all levels of creation."[150]

Llull then establishes a field of six classes of concepts which he called "Principles", each containing nine categories (Figure 5.2). This serves as his "master morphospace" for which he can sort out different sets of "Principles" to interrelate combinatorially. He gives each of the nine categories of each "Principle" a common alphabetic "sign" (at the far left) which he can then use to symbolize them into combinatory configurations. This is what he calls his "Alphabet". The "Divine attributes" ("Figure A": second column) are treated as subjects of predication, while the five following columns are treated as predicates. (Although the *contents* of this "Alphabet" may strike us today as both arbitrary and trivial – which is exactly what Leibniz thought about them some 350 years later – in the theological context of Llull's time they were considered essential.)

Symbols	First figure (Figure A)	Second figure (Figure T)	Questions and rules	Subjects	Virtues	Vices
B	goodness	difference	Whether	God	justice	avarice
C	greatness	concordance	what?	angel	prudence	gluttony
D	eternity or duration	contrariety	of what?	heaven	fortitude	lust
E	power	beginning	why?	man	temperance	pride
F	wisdom	middle	how much?	imaginative	faith	accidie
G	will	end	of what kind?	sensitive	hope	envy
H	virtue	majority	when?	vegetative	charity	ire
I	truth	equality	where?	elementative	patience	lying
K	glory	minority	how? and with what	artifice	pity	inconstancy

Figure 5.2: The "Alphabet" – Llull's master morphospace

5.4 Binary combinatorics

Llull applied combinatorics to his Alphabet in several ways. Firstly, in the simplest case, the First Figure (Figure A") can be combined with itself if one treats it variously as subject and predicate, as shown in Figure 5.4 (below). Llull writes: "This Figure is circular to show that any subject can become a predicate, and vice versa, as when one says, 'goodness is great', 'greatness is good', and so on."[151] (Combinatorial repetition, for instance "BB" or "Goodness is good", is not allowed.)

Figure 5.3: Llull's first "Figure": A (From Bonner, 2007)

Symbols	Figure A Subject	Figure A Predicate
B	goodness	good
C	greatness	great
D	eternity	eternal
E	power	powerful
F	wisdom	wise
G	will	wilful
H	virtue	virtuous
I	truth	truthful
K	glory	glorious

Figure 5.4a: Llull's matrix for Figure A

BC	CD	DE	EF	FG	GH	HI	IK
BD	CE	DF	EG	FH	GI	HK	
BE	CF	DG	EH	FI	GK		
BF	CG	DH	EI	FK			
BG	CH	DI	EK				
BH	CI	DK					
BI	CK						
BK							

Figure 5.4b: The 36 cell combinatory matrix for Figure A as subject-predicate

The possible combinations of nine elements taken two at a time, when neither the inversion of order nor repetitions are allowed, is 36 (Fig. 5.4b).

- **Ternary combinations**

The "Fourth Figure" (Fig. 5.5) consists of three "principles" in combination. Given the 9 variables (B through K), there are 9^3 possible ternary combinations if one allows for repetitions. However, *without* repetitions we get

$$\frac{9!}{3!(9-3)!} = 84$$

ternary combinations (Figure 5.6).

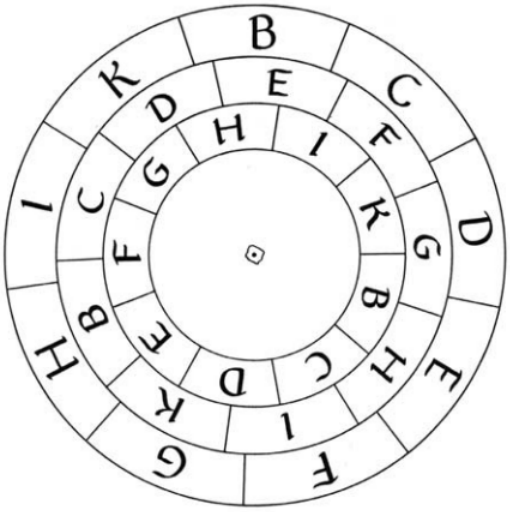


Fig 5.5: Quarta Figura (From Bonner, 2007)

1 BCD	2 BCE	3 BCF	4 BCG	5 BCH	6 BCI	7 BCK	8 BDE	9 BDF	10 BDG	11 BDH	12 BDI
13 BDK	14 BEF	15 BEG	16 BEH	17 BEI	18 BEK	19 BFG	20 BFH	21 BFI	22 BFK	23 BGH	24 BGI
25 BGK	26 BHI	27 BHK	28 BIK	29 CDE	30 CDF	31 CDG	32 CDH	33 CDI	34 CDK	35 CEF	36 CEG
37 CEH	38 CEI	39 CEK	40 CFG	41 CFH	42 CFI	43 CFK	44 CGH	45 CGI	46 CGK	47 CHI	48 CHK
49 CIK	50 DEF	51 DEG	52 DEH	53 DEI	54 DEK	55 DFG	56 DFH	57 DFI	58 DFK	59 DGH	60 DGI
61 DGK	62 DHI	63 DHK	64 DIK	65 EFG	66 EFH	67 EFI	68 EFK	69 EGH	70 EGI	71 EGK	72 EHI
73 EHK	74 EIK	75 FGH	76 FGI	77 FGK	78 FHI	79 FHK	80 FIK	81 GHI	82 GHK	83 GIK	84 HIK

Figure 5.6: Table of ternary combinations (From Bonner, 2007)

Note that using three rings (i.e. variables) lends itself to the formulation syllogistic inference, i.e. a major and minor premise, and a conclusion. Of special interest is the possibility of proving a given conclusion on the basis of a given major premise by finding an appropriate "middle term": one chooses a relationship between the outer and inner rings, and then rotates the middle ring to see if any of the combinations form a valid inference.[152] This was one of Llull's primary goals, to be able to "prove" correct theological doctrine deductively by systematically going through all the combinatoric possibilities.[153]

The number of *three-term* propositions that can be formulated with the "Divine attributes" (Figure A) treated as subjects of predication, and any two of the other five principles as possible predicates, is 7,290. And, for the record, the total number of possible 6-term configurations for the entire morphospace is $\mathbf{9^6}$ **(**=531,441)! Of course, many of these combinations might make little sense, as Llull well understood. It is the responsibility of the "user" to weed out meaningless propositions.

Since we are mainly interested in the epistemological principles involved here, we have only

presented two of the most basic operations in Llull's work, which he developed into a hugely complex system. To get a taste of just how complex the "Art" became, the reader is referred to Antony Bonner's excellent "The Art and Logic of Ramon Llull - A User's Guide" (2007).

Llull also experimented with other ways to represent the interrelationships between Platonic forms, as shown in Figure 5.7.

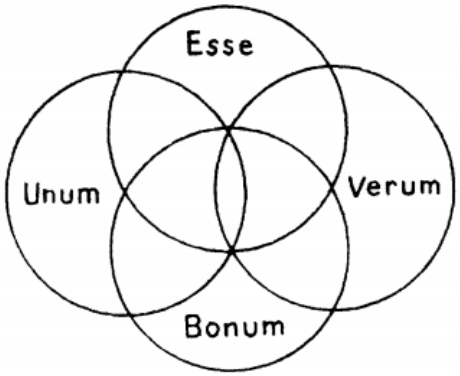


Figure 5.7: Llull's use of a "Euler diagram" for combinatorial blending

This diagram – representing conjunctive forms – is found in a late copy (from 1617) of Llull's "*Opera ea quae ad adinventam*", a work concerning *inventive methods*.[154] It is meant to convey the idea that that nothing exists (Esse) which does not possess unity (Unum), truth (Verum) and goodness (Bonum). (Note that this is not a "full" 4-species (16-fold) typology, since this cannot be represented "completely" using circles.)

Although Llull's applications of combinatoric methods were mainly concerned with theological and philosophical questions, he did apply it to more mundane matters, two examples being medical diagnostics and voting theory. In *Liber Principiorum Medidinae* (1275) he proposed using the combinatory art as a diagnostic system to relate symptoms to treatments[155] – which is what we do today with computerised "expert systems". And in *De Arte Eleccionis* (1299) he developed the earliest known version of a "rank-order-count" voting system[156], which today is generalised in the form of Multi-Criteria Decision Support systems such as the Analytic Hierarchy Process (AHP).

It is interesting to note the different attitudes toward Llull's work by different theological and philosophical traditions of his time. As Frances Yates has noted, Llull's analytic-synthetic "combinatorial art" represented a novel *competing method* to the classical Art of Memory (*ars memoriae*), which was based on the "method of places" (mnemonic *loci*) using mental imagery.[157] In contrast, Llull's art was an early version of a mechanically mediated system of "extended cognition".

> "It is a fact of some historical importance that the two great mediaeval methods, the classical *art of memory* in its mediaeval transformation and the art of Ramon Lull, were both rather particularly associated with the mendicant orders, the one with the Dominicans the other with the Franciscans."[158]

Now it is documented that Llull approached the powerful Dominican order in hopes of getting its support for his method, but without success. At this point in time, the Dominicans had recently refurbished their own methods. In the early 1260's, Albertus Magnus (c. 1200–1280) and his pupil Thomas Aquinas (1225–1274) established their *ratio studiorum*, an educational program aimed at harmonizing Aristotelian teaching with the doctrines of the church, thus initiating the tradition of *Dominican scholastic philosophy*. This established "Aristotelian science" (*ars demonstrandi*) as the approved instrument of Scholastic reasoning.[159]

Llull's *heuristic* program of externalising "place imagery" – thus disburdening internal memory – and analysing Christian doctrine in order to hypothetically generate new premises, which could result in new possible deductive consequences, found little interest among the Dominican establishment. Many of his works were later banned by the Dominican Inquisitor General. The Franciscans, on the other hand, were more interested in his ideas, as their theological orientation – through Augustine, Anslem and Scotus Erigena – was more attuned to the Neoplatonic tradition. Llull eventually became a Franciscan tertiary, a secular member of the Franciscan order.[160]

To sum up: Llull pioneered the use of multivariate, combinatorial "verbal matrices" as a "way of dealing with interrelationships of Platonic forms".[161] This was the beginning of the modern European tradition in scientific method described by Massimo Mugnai (2010) in which:

> "... each complex concept that we conceive is analyzable into all its component parts, down to the most simple ones. Simple, i.e. unanalyzable, concepts are the 'primitives' from which all compound concepts originate. Once one arrives at the primitives, one may choose an alphabet and attribute a sign to each primitive: thus, by combining signs, one will obtain all the complex concepts in a purely mechanical way. *Analysis, synthesis*, and the discovery of an appropriate alphabet were the main stages of [the Llullian] program aiming to renew logic as *both* an art of judgement (the *ars iudicandi*) and an art of discovery (*ars inveniendi*)."[162]

Llull's work has been considered by some as being foundational for Western *combinatorial mathematics* and even for the development of symbolic systems. However, this both overstates and underestimates him. Firstly, Llull's art was not a true *operative symbolic system*: his use of letter-signs was totally abbreviative (see §6). Still worse, each abbreviative "sign" denoted several different attributes at the same time, thus totally defeating the purpose of an operative symbolic system. And although he did draw attention to the mathematical aspect (and got plenty of it wrong), he was far more interested in his Art as a *method* of "inventive logic".[163]

> "One of the principal innovations of the Art was the desire to fuse two fields that had until then been treated separately: the *ars inveniendi* of dialectic and the *ars judicandi* of logic ... and to create a system that could deal with both."[164] ... "Moreover, the combinatory mechanisms with their accompanying graphic devices made possible what was perhaps the most innovative of Llull's accomplishments: to create an Art that was *generative*, which upon a base of a strictly limited number of concepts could build a whole constellation of demonstrations and explanations. And it was this *generative nature* of the Art which held such a powerful fascination for later thinkers."[165]

Llull's basic idea of systematically inter-relating sets of coordinate concepts in order to generate all possible (multivariate) combinations – not only identifying new "blended" concepts *per se*, but even whole configurations – was a major extension of Plato's cross-classificatory modelling. Firstly, it shows that the *synthesis phase* of compositional analysis and synthesis has its own function as an *ars inveniendi*. And, the same system that can generate such combinations can also be turned around to test for their validity (*demonstrandi*).[166] As the eminent Llull scholar Anthony Bonner has observed,

> "... Llull's invention of an *ars combinatoria* as the only possible way of dealing with interrelationships of Platonic forms was to have a considerable impact in the Renaissance ... and would have a decisive influence on Leibniz".[167]

6. Gottfried Leibniz and Constraint-Based Combinatorial Modelling

> "I don't pay much attention to specific discoveries. What I most desire is to perfect the Art of Invention, and to provide methods rather than solutions to problems, for a single method can comprise an infinity of solutions". (Gottfried Leibniz[168])

6.1 The Leibniz Project

Gottfried Wilhelm Leibniz (1646–1716) either needs no introduction or a very substantial one. Due to the nature of this work we are obliged to choose the former alternative.[169] Suffice to say, anyone who has had the opportunity to explore the breadth and depth of his work may arguably conclude that he is the most prolific and *inventive* combination of scientist-philosopher-mathematician in modern history. He may possibly also have the largest *Nachlass* in the history of science, which began to be collated and published in the late 1800s and is still far from complete.[170]

All of his life, Leibniz was concerned with the possibility of developing a universal (symbolic) scientific language which would be capable not only of representing and demonstrating scientific knowledge, but also to serve as a *formal heuristic* capable of facilitating the discovery of new knowledge. Creating adequate symbolic systems in order to accomplish this task is a central aspect of what today is referred to as the "Leibniz project".[171] For this purpose, Leibniz proposed two types of symbolic systems: a universal *symbolic language* (*characteristica universalis*) which could systematically describe and encode knowledge; and a *calculus of reasoning* (*calculus ratiocinator*) as a heuristic tool to conceptually "compute" and formulate new knowledge.

The distinction between these two "systems" – and the theoretical schema of which they are a part – has been studied extensively for the past 50 years.[172] The fundamental difference is this: whereas a "universal symbolic *language*" acts as a closed semantic system within which one can represent cognitive phenomena and operate on concepts in order to solve problems or prove theories, the symbolic *calculus ratiocinator* acts as an open syntactic system which has an inherent generative potential for the reinterpretation and development of new entities and relationships. Leibniz's notion of a *calculus ratiocinator* was the precursor of what Whitehead called the "modern point of view" and what Hintikka (1979) called the "model-theoretical tradition".

> "What [Leibniz] had in mind was a general theory of structures that could provide the syntax for his characteristica universalis. … It would … be the foundation of other branches of mathematics such as geometry, algebra, and the calculus of chances. But most important of all it would be an *instrument of discovery*."[173]

This calculus (or "algebra") of concepts is also basically associated with the notion of *ars combinatoria*, which for Leibniz was a far more general concept than simply a branch of discrete mathematics. It concerns the rules for assembling, combining and generating *formulas in a symbolic language*.[174] For him it was the essence of symbolic modelling:

> "... the art of combinations in particular, as I take it (it can also be called a general characteristic or algebra), is that science in which are treated the forms or formulas of things in general, that is, *quality* in general ... This art is distinct from common algebra, which deals with formulas applied to *quantity* only .. This [quantitative] algebra is thus subordinate to the art of combinations and constantly uses its rules. But these rules of combination are far more general and find application not only in algebra but in the art of deciphering, in various games, in geometry itself when it is treated linearly in the manner of the ancients, and finally, in all matters involving relations of similarity."[175]

Now, as Massimo Mugnai (2018) has noted, the interpreters of Leibniz's logical calculus tend to concentrate on its "demonstrative" or formal deductive aspect. However:

> "As Leibniz explicitly says, the characteristic art should include all kinds of inferences and thus it should not only concern the study of deductive arguments, but also that of inferences based on induction and on premises that are merely probable … [including] its application even for subjects based on *conjectures*.[176]

Indeed, *conjecture* is especially important in those areas in which strict demonstrative inference cannot easily be applied, exemplified by Leibniz as "...in public life, in medicine, in jurisprudence, in military art and in state governing".[177]

6.2 The calculus of concepts and combinatoric modelling

As mentioned above, there are two phases in Leibniz's development of constraint-based *combinatorial modelling*: In his youth – before the "Paris years"– he was greatly influenced by the methods of Ramon Llull. Pombo (2010) notes:

> "… Leibniz recognizes the epistemological value of Lull's project on two important points: a) the status of combinatory as the foundation for the *ars inveniendi*; b) the possibility of applying the same logical procedures to the totality of knowledge, including new knowledge.[178]

Then, from c. 1679 onwards, Leibniz began to develop the symbolic tools to engage conceptual modelling from the "modern point of view". Here we have the rare opportunity of witnessing a one-man watershed: within 20 years he bridged the gap between two scientific epochs. In both cases, he broke new ground. However, once he had transitioned from the pre-symbolic to the symbolic realm, he seems never to have returned to the pre-symbolic version.

> "... Leibniz's greatest difference with respect to Lull concerns the kind of symbolic system able to operate within the calculus. … Leibniz is fully aware of the value of the sign as a *not merely representative* but also a *prospective and heuristic tool*."[179]

In the spirit of *Épistémologie Historique*, we shall present these two "phases" of Leibniz's work *in reverse chronological order*, for the simple reason that the later symbolic treatment critically illuminates, and grounds, the earlier pre-symbolic treatment. Thus we begin with his embryonic *calculus of concepts* and show how this can apply to a simple example of a constrained (Llullian) combinatoric model.

First of all, the important distinction must be made between two "modes" of operative symbolic systems or "calculi": those that are characterized by symbols representing quantitative entities and operations; and those characterized by symbols representing non-quantified entities and operations. That is, the difference between "computing with numbers" and "computing with concepts".[180] Leibniz was quite explicit about this:

> "There are other completely different kinds of analysis or combinations, about the order of things, about qualities, actions and an infinity of other things, which have to be expressed in completely different ways (but through letters or other simple signs) other than quantities…".[181]

Leibniz explored and developed both *quantified* and *non-quantified* calculi, the most famous example of the former being his differential and integral calculus. In the latter case, he was interested in a symbolic system of pure *concepts*. Such a *general characteristic*, or *general algebra*, is what – by the end of the 19th Century – would be called Universal Algebra. And, as noted earlier, the first to write a comprehensive treatise on Universal Algebra was A. N. Whitehead (1998). In it he explains:

> "Such algebras are not essentially concerned with number or quantity; and this bold extension beyond the traditional domain of pure quantity forms their peculiar interest."[182]

- **The axioms**

In a series of essays written between 1679 and 1690, Leibniz put forward sets of axioms and operations for this "general combinatorial algebra" which are regarded as precursors to 19th century algebraic logic. The general history of this development has been well covered in the literature.[183] Our particular interest here concerns how certain of these axioms and operational rules retrospectively satisfy the procedures involved in Leibniz's earlier (non-symbolic) constraint-based combinatorial modelling. Central to this is his formulation of a "concept operator" which he calls *real addition* and designates with the symbol $\oplus$.[184] Swoyer (1994) calls $\oplus$ a "conjunction-like operator" for forming compound concepts:

> "Although Leibniz stresses that his calculus is amenable to alternative interpretations, he very frequently interprets its characters as signifying *concepts* and real addition as an operation for *conjoining* them".[185]

Here are the relevant axioms (in simplified notation) that Leibniz formulates for *real addition*, which are central to his earlier combinatoric modelling:[186]

(1) $A \oplus A = A$ (Idempotence)
(2) $A \oplus B = B \oplus A$ (Commutativity)
(3) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ (Associativity)

(1) **Idempotence** ("same strength" or "same value") means that a concept added to or conjoined with itself remains unchanged. For instance, the *concept of a horse* "plus" the *concept of a horse* is neither the concept of two horses nor two concepts of a horse. It is

simply the *concept of a horse*. We see this property expressed e.g. in "cross-impact analysis" where a single factor—if we do not want to go into its internal workings – is not regarded as having a cross-impact on itself. This is also the case for *cross-consistency assessment* in morphological and other associated matrix models.

(2) **Commutativity** implies that conjoined concepts mean the same thing whichever their order, i.e. we are concerned here with *combinations*, not with *permutations*. Of course, one could formulate an axiomatic system in which (non-communicative) permutations are considered meaningful, in which case the rule would be $A \oplus B \neq B \oplus A$. But that would be another type of "system", e.g. (causally directed) cross-impact analysis.

(3) **Associativity** concerns how operators can be grouped with the same precedence in the absence of parentheses. Thus (3) can be written without parentheses as: $A \oplus B \oplus C$. Indeed, any string or chains of single concepts, or variables, can be formulated in this way, each additional variable forming a new "dimension".[187]

Note that $(A \oplus B)$ can be *interpreted* both as an *extensional* (set theoretical) intersection or as *intentional* concept conjunction. Leibniz usually employs the *intentional* interpretation.[188]

The "calculus of real addition" can thus represent *coordinate concept-conjunction* and the combinatorial matrix thus produced. For instance, while $A \times B$ usually denotes a Cartesian Product, i.e. a set of *ordered pairs* forming the "cross-order" of two *numerical* variables, in "real addition" $A \oplus B$ can denote a set of *non-ordered* pairs representing the fusion or integration of *concepts*. Thus if the objects of this operation are *tuples* or *category variables*, then "real addition" produces a (discrete) *combinatorial coordinate matrix*, a.k.a. an N-dimensional typological field or morphospace (see below).

Here we see that "chiastic order", Leibniz's *ars combinatoria* and Whitehead's "science of cross-classification" are essentially concerned with the same type of structural reasoning. And just as Whitehead notes that his "science of cross-classification" is concerned with "... *any subject matter to which the formal axioms may apply*"[189], Leibniz likewise states that:

> "As general algebra [*speciosa generalis*] is merely the representation and treatment of combinations by signs, and as various laws of combination can be discovered, the result of this is that various methods of computation arise. ... Consequently, *whenever these laws* [or axioms] *are observed, the present calculus can be applied*."[190]

6.3 A simple constraint-based combinatorial model

One of Leibniz's clearest examples of a "subject matter" that follows the rules of *real addition* is given in a metaphysical exercise. Now, Leibniz considered our ("actual") universe to be an optimal architectonic system, not only in terms of systematic fitness (optimisation and harmonisation), but also in terms of plenitude. Thus the actual world is a maximization of its possible alternative contents.[191] In his "Dialogue between Theophile and Polidore"

(1679)[192], he describes how this sort of ontological "possibility maximization" would work itself out combinatorially in a both *balanced* and *maximally bounteous* ecosystem. This example serves both to illuminate a metaphysical principle, and to expound certain methodological principles involved in constrained combinatorial modelling and inference by exclusion.

Leibniz begins by positing a set of "beings", each representing a particular quality or property that could be present in a possible world. He then introduces a number of specific constraints or "incompatibilities" between certain pairs of the properties:

> "Let it be that there are seven possible beings ABCDEFG, all equally perfect and with equal claims to existence, and that the following six *incompatibilities* obtain: A [is incompatible with] with B, B with D, D with G, G with C, C with F, and F with E. Then [in consequence of these constraints,] two [beings] could be made to exist together in fifteen ways: AC, AD, AE, AF, AG, BC, BE, BF, BG, CD, CE, DE, DF, EG, FG. Again, three of them [at a time] could be realizable as follows: ACD, ACE, ADE, ADF, AEG, AFG, BCE, BEC, and BFG. But four [of them at a time] could be realized in only one single way ACDE. It is this that would be chosen among all the other possibilities, since it is this that one obtains the *most possible*. In consequence, this quartet ACDE would exist in preference to [any other of] the rest."[193]

Now if we relate each of the N=7 properties to each other, then N×N denotes its "Cartesian product" (or cross-impact matrix) as in Fig. 6.1a. However, applying instead N⊕N we can see the effects of the "axiomatic rules" that govern Leibniz's notion of "real (concept) addition" (Fig. 6.1b). We first apply the constraints given by the "axioms", and then by Leibniz's six specific "incompatibilities".

	A	B	C	D	E	F	G
A	AA	BA	CA	DA	EA	FA	GA
B	AB	BB	CB	DB	EB	FB	GB
C	AC	BC	CC	DC	EC	FC	GC
D	AD	BD	CD	DD	ED	FD	GD
E	AE	BE	CE	DE	EE	FE	GE
F	AF	BF	CF	DF	EF	FF	GF
G	AG	BG	CG	DG	EG	FG	GG

Figure 6.1a: NxN dyadic relationships

	A	B	C	D	E	F	G
A	O	-	-	-	-	-	-
B	X	O	-	-	-	-	-
C	AC	BC	O	-	-	-	-
D	AD	X	CD	O	-	-	-
E	AE	BE	CE	DE	O	-	-
F	AF	BF	X	DF	X	O	-
G	AG	BG	X	X	EG	FG	O

Fig. 6.1b: Constrained N⊕N matrix

(1) The "O" marked cells (Fig. 6.1b) along the diagonal are *repetitions* or self-conjunctions. They are thus disregarded following **Axiom 1**: Idempotence (A⊕A = A).

(2) The "*off-diagonal upper-right hemimatrix*" (ODURH – i.e. the cells marked with hyphens "-") contains mirror image permutations of the combinations in the "*off-diagonal lower-left hemimatrix*" (ODLLH, the white & X-marked cells), and is thus disregarded following **Axiom 2**: Commutativity (A⊕B = B⊕A).

We are left with the ODLLH. The number of pair-wise relationships in this hemimatrix is ½N(N-1) which represents the *triangular number series* and is indicative of dyadic combinatorial relationships.

(3) Finally, the X-marked (red) cells represent Leibniz's stipulated *pair-wise constraints*. The remaining (white) cells are the 15 dyadic relationships that survive the constraints. Figure 6.2 shows the *meet-lattice*.[194]

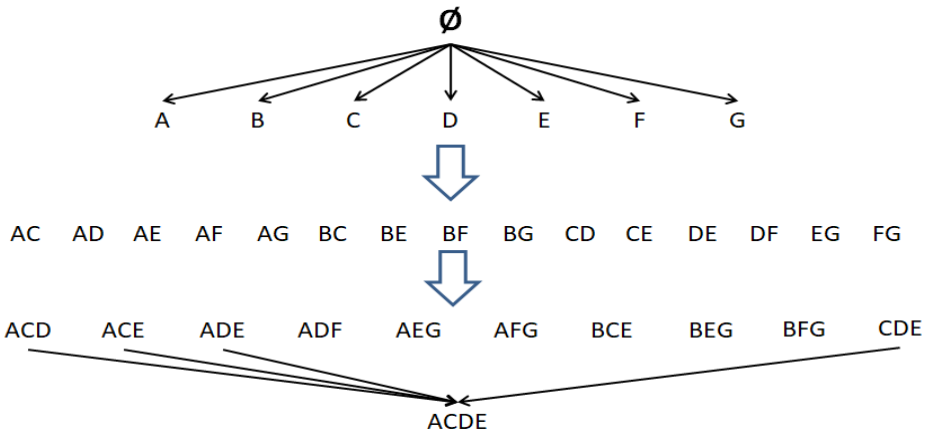


Figure 6.2: The constrained meet-lattice

From this formal introduction we can now give an example of a full-blown, fully modern constraint-based combinatorial model developed by Leibniz in 1666.

6.4 Dissertatio de arte combinatoria (1666)

The full (English) title of *De Arte Combinatoria* (DAC) is:

> "*A Dissertation on Combinatorial Art*, in which starting from the foundations of Arithmetic, a theory of *Complications* and *Permutations* is constructed out of new rules, their use across the entire field of the sciences is demonstrated, and new seeds of the Art of Heuristic, or the *Logic of Discovery* are sown."[195]

The DAC was an extension to Leibniz's doctoral thesis written when he was 19 years old. In it, he describes – *inter alia* – the principles and process involved in discrete variable (de-compositional/re-compositional) combinatoric modelling. For our purposes, the most interesting example that he presents is that of identifying all of the possible (valid) syllogistic forms by modelling the conceptual entities and relational rules involved in the *categorical syllogism*. We can systematically demonstrate how Leibniz did this with the aid of dedicated morphological modelling software.[196] First some background.

The *syllogistic system* that Leibniz is treating here is an extension of Aristotle's original categorical syllogism (introduced in *Prior Analytics*), which was further developed within the medieval (Scholastic) tradition. It is a complex system and we do not wish to get bogged down in all of its details – nor is this necessary in order to demonstrate the *principles* of Leibniz's constraint-based combinatoric approach.

Note also that the *results* which Leibniz produced concerning all possible valid syllogistic forms were not, in themselves, new. This had already been systemised (mainly by "brute force") at least a century earlier by Johannes Wirth (a.k.a. Hospinianus, 1515–1575). What Leibniz wants to show is that this result can be obtained more easily *algorithmically*, by modelling the problem's variables and applying its constraining rules. Thus one can treat Leibniz's application of this as a "product sample" to demonstrate how combinatoric modelling makes problems like this not only easier to solve, but also more transparent.[197]

We start with Aristotle's original version of the syllogism in order to more easily demonstrate the modelling principles involved. The *categorical syllogism* is a system of logical arguments consisting of three subject-predicate *propositions* that combine to produce formal inference. For instance, for the syllogism called *BARBARA* the propositions are:

• Major premise	e.g. All humans are mortal
• Minor premise	e.g. All Greeks are human
[∴= *therefore***]** Conclusion....	∴ All Greeks are mortal

- **Quantity and Quality**

The "mood" of a syllogism is defined by the combination of two things, termed ***Quantity*** and ***Quality*** which – as discussed in §2 *supra* – make up the two "dimensions" of the Aristotelian logical "Square of opposition".

The **Quantity** of a proposition refers not to numerical entities, but to "logical operators". It concerns whether a proposition concerns "all" of something, or "some" of something.

Universal (**U**): **All A** is **B**
Particular (**P**): **Some A** is **B**

Since a syllogism is determined on the basis of **quantity** for *each* of the three propositions (each of them either U or P) then the number of possible combinations is $2^3 = 8$ (Fig. 6.3).

Quantity		
Major Premise	**Minor Premise**	**Conclusion**
U	U	U
P	P	P

Figure 6.3: Morphospace of the three propositions' possible "quantities"

The **Quality** of a proposition concerns whether it is affirmative (**A**) or negative (**N**), i.e. does it state that "something IS something", or that "something IS NOT something". Since this binary distinction also ranges over all three propositions, then there are likewise $2^3 = 8$ possible *quality variations* for a syllogism.

Linking **Quantity** and **Quality** together, there are $2^3 \times 2^3 = 64$ possible (formal) syllogistic configurations (Fig. 6.4)

Quantity			Quality		
Major	**Minor**	**Conclusion**	**Major**	**Minor**	**Conclusion**
U	U	U	A	A	A
P	P	P	N	N	N

Figure 6.4: The 6 binary-variable morphospace of the Aristotelian categorical syllogism, with one "mood" shown called *Camestros,* designated (UA,UN,PN).

Now, there are certain "rules" which place constraints on how **quantities** and **qualities** can and cannot combine in order to produce valid inference: two rules for **quantity** and three for **quality**. (Note that what is termed the "weaker" premise in **quantity** is the *particular*, and in **quality** the *negative*.)

Constraints on Quantity (U/P) :

1. From two **particular** premises (**P**), nothing follows.
2. The conclusion follows the **quantity** of the *weaker* of the two premises.

Constraints on Quality (A/N):

3. From two **negative** premises (**N**), nothing follows
4. The conclusion follows the **quality** of the *weaker* of the two premises.
5. A **negative** (**N**) conclusion cannot follow from two **affirmative** premises (**A**).

These rules can now be applied employing a "cross-consistency matrix" which represents all of the pair-wise relationships in the (syllogistic) morphospace in Fig. 6.4. (As can be seen, following the rules of Leibniz's "real addition", this matrix is the "off-diagonal, lower left hemimatrix" of the variables of a 6x6 matrix, although in this case each "cell" in the matrix is now a 2x2 compound of U/P and A/N combinations.)

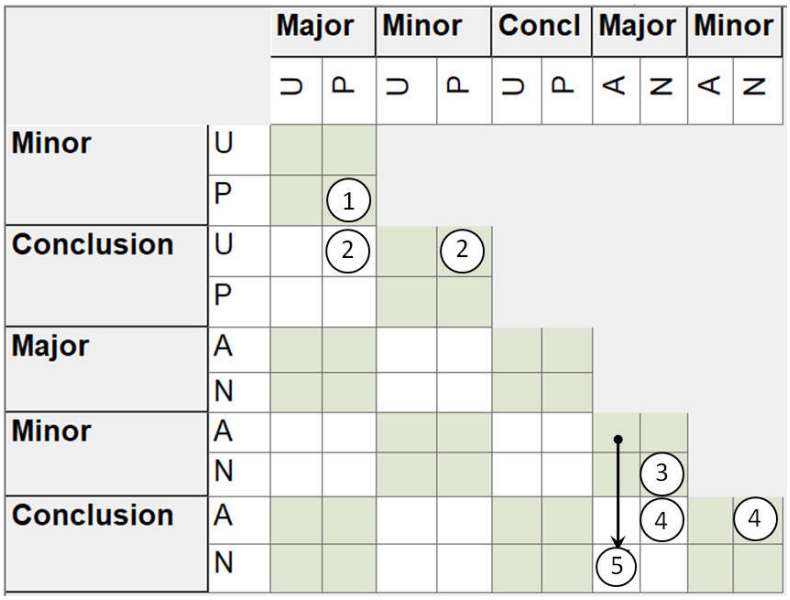


Figure 6.5: Cross-consistency (hemi-)matrix of the morphospace in Fig. 6.4.

Applying constraining rules 1 and 2 to **Quantity** reduces the number of valid alternatives to **4**: UUU, UUP, PUP and UPP. Applying the similar rules 3 and 4 to **Quality** likewise reduces the number of valid alternatives to **4**: AAA, NAN, ANN and AAN. However, rule 5 – a three-part constraint – disallows a Negative conclusion resulting from two Affirmative premises, thus deleting AAN, leaving **3**: AAA, NAN, ANN. Thus the total number of valid moods is the product of (**4** possible quantitative alternatives) x (**3** possible qualitative alternatives) = 12.

	Major,Minor,Concl.	AEIO	Name
1	**UA, UA, UA**	**AAA**	Barbara
2	**UN, UA, UN**	**EAE**	Celarent
3	**UA, UN, UN**	**AEE**	Camestres
4	**UA, UA, PA**	**AAI**	Barbari
5	**UN, UA, PN**	**EAO**	Celaro
6	**UA, UN, PN**	**AEO**	Camestros
7	**UA, PA, PA**	**AII**	Darii
8	**UN, PA, PN**	**EIO**	Ferio
9	**UA, PN, PN**	**AOO**	Baroco
10	**PA, UA, PA**	**IAI**	Diabis
11	**PN, UA, PN**	**OAO**	Bocardo
12	**PA, UN, PN**	**[IEO]***	Frisesmo

Notation type		
UP/AN	**AEIO**	**Reads**
UA =	**A**	All X is Y
UN =	**E**	No X is Y
PA =	**I**	Some X is Y
PN =	**O**	Some X is not Y

Figure 6.6: 12 valid moods in UP/AN and AEIO designation

Note that mood #12 – the so-called "anomalous mood" Frisesmo (IEO) – is a special case which is usually deleted from the list of valid moods because it does not formally belong to any of the four so-called "figures" (which concerns the placement of the middle term in the premises, i.e. whether it appears as subject or predicate). However, even if Frisesmo is formally "homeless", it is no less a *valid syllogistic form*.[198] Our "model" does not allow for such extraordinary exclusions, but simply returns the results that the constraining rules formally imply.

Figures 6.7 and 6.8 show the computer generated list of the 12 syllogistic moods with two example configurations.

Configuration List - Arist - Valid Moods

1	**AAA**	**Barbara**
2	EAE	Celarent
3	AEE	Camestres
4	AAI	Barbari
5	EAO	Celaro
6	AEO	Camestros
7	AII	Darii
8	EIO	Ferio
9	AOO	Baroco
10	IAI	Diabis
11	OAO	Bocardo
12	[IEO]	Frisesmo

Quantity			Quality		
Major Premise	Minor Premise	Conclusion	Major Premise	Minor Premise	Conclusion
ALL (U)	ALL (U)	ALL (U)	Affirmative	Affirmative	Affirmative
SOME (P)	SOME (P)	SOME (P)	Negative	Negative	Negative

Fig 6.7: BARBARA

Configuration List - Arist - Valid Moods

1	AAA	Barbara
2	EAE	Celarent
3	AEE	Camestres
4	AAI	Barbari
5	EAO	Celaro
6	AEO	Camestros
7	**AII**	**Darii**
8	EIO	Ferio
9	AOO	Baroco
10	IAI	Diabis
11	OAO	Bocardo
12	[IEO]	Frisesmo

Quantity			Quality		
Major Premise	Minor Premise	Conclusion	Major Premise	Minor Premise	Conclusion
ALL (U)	ALL (U)	ALL (U)	Affirmative	Affirmative	Affirmative
SOME (P)	SOME (P)	SOME (P)	Negative	Negative	Negative

Fig. 6.8: DARII

- **Extended version**

In the DAC, Leibniz takes on the extended version of the syllogistic. In this version **quantity** is extended by two new types: called *Indefinite* and *Singular*:

Universal (**U**) -> **All X** is Y
Particular (**P**) -> **Some X** is Y
Indefinite (**I**) -> **X is Y (i.e. X is** either **All Y** or **Some Y)**
Singular (**S**) -> A **Single instance** of **X** (rather than the **class X**) is **Y**

Thus the total number of "formal" configurations is now (4*4*4) * (2*2*2) = 512 (Fig. 6.9). This results in *eight times* as many configurations as in the basic Aristotelian case (= 64).

Quantity			Quality		
Major Premise	Minor Premise	Conclusion	Major Premise	Minor Premise	Conclusion
U	U	U	A	A	A
P	P	P	N	N	N
I	I	I			
S	S	S			

Figure 6.9: Total extended syllogistic problem space containing 512 formal moods.

As concerns the constraining rules for **quantity**, Leibniz treats the **Singular** (S) the same as the **Universal** (U), and the **Indefinite** (I) the same is the **Particular** (P). Fig. 6.10b shows how these rules are then applied to the extended **quantities** (where X marks a constraint).

Major	Minor	Conclusion
U	U	U
P	P	P
S	S	S
I	I	I

Figure 6.10a: Extended quantities

		Major				Minor			
		U	P	S	I	U	P	S	I
Minor	U								
	P		X		X				
	S								
	I		X		X				
Conclusion	U		X		X		X		X
	P								
	S		X		X		X		X
	I								

Figure 6.10b: Excluded quantity combinations

This results in 32 configurations for "quantity (Fig. 6.11), again *eight times* as many as in the original Aristotelian version.

U, U, U	**S, S, S**	**U, U, S**	**U, S, U**	**S, U, U**	**S, S, U**	**S, U, S**	**U, S, S**
U, U, P	**U, U, I**	**S, S, P**	**S, S, I**	**U, S, I**	**U, S, P**	**S, U, I**	**S, U, P**
U, I, I	**U, P, P**	**U, P, I**	**U, I, P**	**S, I, I**	**S, P, P**	**S, P, I**	**S, I, P**
I, U, I	**P, U, P**	**I, U, P**	**P, U, I**	**I, S, I**	**P, S, P**	**I, S, P**	**P, S, I**

Figure 6.11: The 32 (8x4) combinations of "quantity"[199]

Now we already know that there are only three valid **qualities**: AAA, NAN, ANN. This is the same for both the extended and original version of the syllogistic. Adding these to the

quantity morphospace gives (3 x 32) = 96 distinct valid moods (Fig. 6.12). Finally, if from these 96 moods one wants to exclude the *eight* that are associated with the "homeless" mood *Frisesmo*, there remain 88, i.e. the agreed upon number of "valid moods". ***QED***

Major	Minor	Conclusion	Valid Quality
U	U	U	AAA
P	P	P	NAN
S	S	S	ANN
I	I	I	

Figure 6.12: Morphospace of the extended version of the syllogistic

As Mugnai et. al. (2020) note, Leibniz approached the combinatorial derivation of the valid syllogistic forms in a somewhat different and more round-about way. He constructs and constrains the moods in other bits and pieces and then stitches them together. Part of the reason for this may have been due to the way in which the syllogistic had been dealt with historically, and Leibniz is simply following this path in his combinatorial treatment. However, another reason is that Leibniz didn't have access to a modern digital computer and dedicated software; i.e. he could not simply formulate the total problem space, enter the constraining rules on a cross-consistency matrix, and then push a button.

* * *

In the 18th century, the Llullian (non-symbolic) method of discrete variable combinatorial modelling, although not totally disappearing, more or less stagnated. There were, I think, two main reasons for this; one external and one (methodologically) internal.

On the external side, the 17th century "symbolic revolution" totally overshadowed the (pre-symbolic) Llullian method of combinatorial modelling. Ironically, Leibniz was involved both in completing the Llullian program and then dooming it to obscurity. His infinitesimal calculus would dominate the scene over the next two centuries and open the door to partial differential equations, the calculus of variations (and the Principle of Least Action), generalised coordinates and – as the crowning achievement to all of this – a *rational mechanics*. In this historical context, Llullian combinatorial modelling was totally eclipsed. (However, as we shall see, such modelling nonetheless had – and still has – an *inventive potential* in precisely those "conjectural" areas of application which Leibniz exemplified as "...in public life, in medicine, in jurisprudence, in military art and in state governing".)

The “internal” (methodological) hinder concerns what is called *combinatorial explosion*. The number of formal configurations in a morphospace increases exponentially (or factorially) with the number of variables (tuples) involved. As noted in §5, Llull’s master morphospace consists of six 9-tuples, creating 9^6 = 531,441 (formal) configurations, far too many to deal with “by hand”. Right up until the late 20th century, this was one of the major hinders to working with multivariate combinatorial modelling. Thus, the next significant step in its evolution would have to wait some 300 years, until it was provided with dedicated computer support.

7. Fritz Zwicky and the Morphological Approach

7.1 20th Century background - again[200]

The introduction to this work began by examining A. N. Whitehead's involvement in the early 20th century "Leibnizian revival". Now we have returned to this very period. The said revival – principally taking place in Germany and France – was fuelled by the works of two prominent scholars who had accessed the huge Leibnizian *Nachlass* in Hanover – the "Hannover manuscripts".[201] The first was the German mathematician and mathematical historian Carl Immanuel Gerhardt (1816–1899) who produced two major collections of Leibniz's works on mathematics and philosophy: the 7-volume *Mathematische Schriften* (collated and published between 1849–1863) where hitherto unknown mathematical material was discovered; and the later work *Die philosophischen Schriften* publish from 1875 to 1889.

Following this, the French mathematician and logician Louis Couturat (1868–1914) further developed Gerhardt's work with two publications: *La Logique de Leibniz* (1901) and *Opuscules et Fragments Inedits de Leibniz* (1903).[202] These studies included an account of Leibniz's doctorial thesis extension "De arte combinatoria", with its description of conceptual combinatorial analysis and synthesis.[203]

Furthermore, two important figures in modern philosophy of science, Henri Poincaré (1854-1912) and Wilhelm Ostwald (1853–1932), were advancing Leibnizian *ars combinatoria* as the basis for the "discovery process" (*ars inveniendi*). Poincaré, who was well acquainted with the work of Gerhardt and Couturat, and was involved in preparing a new international edition of Leibniz's works, described the creative process in terms of an "unconscious combinatory activity"[204]

.

> "... the useful or interesting combinations are those which reveal to us unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another."[205]

Ostwald, Nobel Prize winner in chemistry, philosopher of science and crusader for the reform of scientific education, promoted *consciously explicit* creative concept conjunction and conceptual integration:

> "... we can easily make manifold arbitrary combinations of concepts from different experiences, since our memory freely places them at our disposal, and from such a combination we can form a new concept".[206]

It is also interesting to note that Ostwald, Poincare and Whitehead were all active in promoting a heuristic *historical perspective* in scientific education. Poincaré was also one of the precursors of the *Épistémologie Historique* movement in France.

However, the actual practical application of the Llullian-Leibnizian method of constraint-based combinatoric modelling would eventually come from a totally different source: from the maverick astrophysicist, rocket scientist and conceptual innovator Fritz Zwicky (1898–

1974). In the 1940's Zwicky begin to systematically apply combinatorial heuristic modelling to real-world issues within several different fields of enquiry, including astrophysics, engineering design, policy analysis and social-ethical issues. He called it variously the *morphological approach*, *morphological analysis* and (intriguingly) *morphologischen Feldüberdeckung* (morphological *field conjunction*). Today, in its computer-aided version, this is known as General Morphological Analysis (GMA).[207] Writes Zwicky in 1969:

> "Attention has been called to the fact that the term morphology has long been used in many fields of science to designate research on structural interrelations – for instance in anatomy, geology, botany and biology. ... I have proposed to *generalize* and *systematize* the concept of morphological research and include not only the study of the shapes of geometrical, geological, biological, and generally material structures, but also to study the more abstract structural interrelations among phenomena, concepts, and ideas, whatever their character might be."[208]

There is somewhat of a mystery as concerns Zwicky's actual source of inspiration in developing and applying this method, an issue which is beyond the scope of the present work. Suffice to say, there is nowhere in his published works (as far as I can ascertain) any reference to Ramon Llull or Gottfried Leibniz. Zwicky declared that he was inspired by Johann von Goethe's notion of *morphology* (whence the "morphological approach").[209] Now, although Goethe's morphological method does not explicitly employ constraint-based combinatorial modelling, it certainly does allude to the idea of a morphospace. For instance, in text now collected under the title of *Zur Morphologie* from 1795, Goethe writes:

> "… here is a proposal for an anatomical archetype [Typus], a general image [Bilde] in which the forms [Gestalten] of all possible animals are contained, and according to which each animal can be described in a certain orderly way. (Goethe, 1820, p. 23. My translation and brackets.)

In a letter to Herder concerning "plant life", he expounds the same theme:

> "With such a model … one will be able to contrive an infinite variety of plants. They will be strictly logical plants – in other words, even though they may not actually exist, they could exist." (Cited in Magnus, 1961, p.45).

Petteni (2022) comes even closer to describing a modern morphological model when she portrays Goethe's notion of an "anatomical archetype" or *Urtyp* as:

> "… a computational device explicitly designed to formulate open-ended classification rules … in order to facilitate the *definition of natural domains*, and, in a subsequent step, to explore the invisible *model space of such domains*" (p. 45); and "... a *structural matrix* enabling the exploration of nature's virtual realm." (p. 51). (Emphasis added)

Fritz Zwicky is not a household name in science today. He was not a superstar of the likes of Einstein, Hubble or Oppenheimer. Yet his influence was significant; far more than his present-day lack of recognition would suggest. He was one of the broadest and most inventive scientists of his time, and combined "high-flyer" theoretical studies with eminently practical, humanitarian activities.

Zwicky was born in Varna, Bulgaria (1898) to a Swiss father (merchant and diplomat) and ethnic Czech mother.[210] At the age of six, he was sent to his father's ancestral district Glarus (Switzerland) in the middle of the Schwyzer Alps (Zwicky was a life-long mountain climber). Although expected to take up a career in commerce, his early bent for science persuaded his father to allow him to study engineering instead. In 1914 he moved to Zurich where he enrolled in the Swiss Federal Institute of Technology (ETH). There he switched to mathematics and experimental physics, wrote his examination essay for no one less than Herman Weyl, and in 1922 took his doctorate with a dissertation on ionic crystals. Three years later he was invited to study in the U.S. and ended up at the California Institute of Technology (Caltech) in Pasadena. There he initially worked with the great experimental physicist Robert Millikan (who once said that he kept Zwicky around because some of his hair-brained ideas might turn out to be right – which many of them did!).

From this point on, Zwicky more or less worked out of Pasadena, both as a faculty member of Caltech (1927–68) and research director/consultant for Aerojet Engineering Corporation (1943–61). He became Professor of Astrophysics at Caltech in 1942 and was a member of the staff of Mount Wilson and Palomar Observatories until his retirement in 1968.

Zwicky is primarily known for his work in astrophysics, and especially his comprehensive galaxy surveys. However, he thrived on investigating and theorizing about extreme phenomena and boundary conditions. This led him both to develop new methods for systematically investigating multi-dimensional problem complexes and to formulating a number of hypotheses which represented significant breakthroughs in astronomy.

Zwicky and Caltech colleague Walter Baade were the driving forces behind acquiring and installing the first Schmidt telescope to be used in a mountain-top observatory – the famous 18-inch Palomar Schmidt – in 1935. Bernhard Schmidt's revolutionary new telescope made it possible to photograph large areas of the sky quickly, with little distortion. Zwicky used it to make the first rapid survey of the heavens, mapping out thousands of galaxies (called the Zwicky Galaxy Database). As a result of this, Zwicky discovered that galaxies tended to cluster, opening up a new chapter in the history of astronomy and cosmology. At the same time, he applied the *virial theorem* to the Coma cluster of galaxies and obtained evidence of *dunkle Materie*, thus starting off the debate on dark matter. (The 18-inch Schmidt was later used by Gene and Carolyn Schumacher to discover the comet Schumacher-Levy 9, which smashed into Jupiter in July 1994.)

Pursuing the idea that "bright novae" were of fundamental interest for determining the distance to far-off galaxies, he and Walter Baade coined the term *supernova*. These, Zwicky proposed, marked the transition from ordinary stars to neutron stars – which he was the first to hypothesize – and were the origin of cosmic rays. This was an amazing (and correct) triple hypothesis and was an important step in the still on-going project to determine the size and age of the (visible) universe. (His neutron-star-hypothesis entered mainstream astronomy in the 1960's.) In 1937 Zwicky proposed that galaxies could act as gravitational lenses.[211]

Besides numerous other contributions to astrophysics, Zwicky was active in the aerospace industry. In the aftermath of WWII, he was appointed head of the U.S. Air Force teams that went to Germany and Japan to evaluate wartime research on jet and rocket propulsion. He was subsequently awarded the Medal of Freedom by President Truman for his work. He was also research director at Aerojet Engineering and was involved in the development of jet and rocket propulsion systems – for which he obtained a number of patents.

The importance that Zwicky placed on the practical applications of morphological modelling is evident: When, in 1948, he was invited to deliver the prestigious Oxford University Halley Lecture, he gave it the title "Morphological Astronomy" and built his presentation around his "morphological method", giving examples of how it could be applied for future astronomical studies both technically and pedagogically. He stated that he had tackled almost every single problem from the perspective of morphology because *quality precedes quantity*:

> "Morphology ... concerns itself with the qualitative patterns of all things. Quantitative aspects enter on the secondary plane when the decision has to be made which of the possibilities brought to light by the basic morphological analysis best serves a desired purpose."[212]

7.2 Morphological models and "wicked problems"[213]

Whether or not Zwicky was aware of the earlier work of Leibniz, he is working in the same spirit and with the same types of problems. As discussed earlier, Leibniz pointed out the importance of combinatorial heuristic modelling not only in "natural philosophy", but also "...in public life, in medicine, in jurisprudence, in military art and in state governing".[214] Zwicky likewise states:

> "In physics, biology, botany, geology, and other disciplines the morphological method has played an important role. In *sociology* and *statecraft*, where prejudices, conventions and narrow ideologies interfere, the method is more difficult of application although is it precisely in these fields where it could be most beneficial."[215]

Such areas of application differ fundamentally from those of natural science and engineering design *per se*. The main difference is that they are essentially *policy driven problems* involving values, norms, goals and motivations. This is the type of problem that Zwicky addressed in an article concerning the "smog problem" in Los Angeles in the late 1950's. There he states that the morphological approach is especially applicable to such complex policy driven problems that require:

> "… an integrated view which relates [technical,] political, psychological and ethical factors ...[which] … add up to a complex task which is beyond the power of ordinary scientific, technical and managerial experts."[216]

A decade later Horst Rittel, the design theorist at the University of Berkeley in California, employed the term "wicked problems" to refer to such complex, policy driven issues.[217] Rittel and colleague Melvin Webber then introduced this concept to a broader audience with their article for *Policy Sciences*: "Dilemmas in a General Theory of Planning" (1973). Both the article and the concept are now famous, at least within the design community (see e.g.

Buchanan, 1992), and with the "soft side" of Operational Research/Management Science (OR/MS). Since morphological modelling applied to policy driven issues essentially deals with "wicked problems", a summary of the concept is in order here.

The notion of *wicked problems* (WP) is a *term d'art* in Operational Research & Management Science used to designate sociotechnical issues which are ill-defined, ambiguous and associated with strong moral, political and professional issues. They concern competing stakeholder interests, rival ideologies and any number of subjective "people problems". Most importantly, wicked problems won't keep still: they are open-ended sets of complex, interacting issues continuously evolving in a dynamic social context. Often, new forms of wicked problems emerge *as a result* of trying to understand and "solve" one of them.[218] This is why modelling such problems require *modal* – i.e. *alethic* and *deontic* – considerations.

Of course, *wicked problems* are nothing new; they are as old as human society. But they were raised to a new level of consciousness – and given a catchy name – in the context of post war urban society of the 1950's and 60's. Rittel & Webber put forward ten example "criteria" for identifying WPs. We need not go through all of these. For our purposes, we present those aspects of WPs to which GMA has been found to be especially applicable.[219]

Criterion #1: "... in order to describe a wicked *problem* in sufficient detail, one has to develop an exhaustive inventory for all the conceivable *solutions* ahead of time."

Performed properly, GMA results in an inference model which strives to represent the total problem space, and as many of the potential solutions, to the given problem complex as possible. One literally builds up an inventory of all possible solution concepts, in order to better understand the actual problem. This goes a long way in satisfying this seemingly incredible criterion.

Criterion #3: Different stakeholders …"are likely to differ widely to accord with their group or personal interests, their special value-sets, and their ideological predilections."

The process of creating morphological inference models through facilitated group workshops is as least as important as the end-product - i.e. the model itself. As many stakeholders as possible should be engaged in the work, in order to create a common terminology, common problem concept and common modelling framework. Principal stakeholders and subject specialists should therefore be brought together in a series of workshops to collectively 1) structure as much of the problem space as possible, 2) synthesize solution spaces, 3) explore multiple solutions on the basis of different drivers and interests and 4) analyze stakeholder structures. The different stakeholders do not have to agree on a single, common solution, but must be encouraged to understand each other's positions and contexts.

This last point is crucial. Consensus means "general agreement or concord" within a group. However, one needs to differentiate between first-order and second-order consensus. The normal first-order form is that of gaining a common standpoint or agreeing upon a common solution. This is seldom the case with stakeholder groups working with wicked problems. Second order consensus is when stakeholders in a group learn to accept each other's specific stakeholder positions - on the basis of understanding their different "stakeholder reasons" for these positions.

Criterion #7: "...part of the art of dealing with wicked problems is the art of not knowing too early which type of solution to apply."

In GMA we call this "remaining in the mess", i.e. keeping one's options open long enough to explore as many relationships in the total problem space (morphospace) as possible, before starting to formulate solutions. This can be a frustrating process for inveterate "problem solvers", but is an absolutely necessary procedure when modelling wicked problems.

Criterion #8: "Every wicked problem can be seen as a symptom of another [wicked] problem."

With a morphological inference model, one can treat any particular parameter or "issue" as the starting point, driver or "independent" variable. This allows one to change perspectives and treat different issues as both causes and effects. Everything is connected, which is what both wicked problems and GMA is all about.

Criterion #10: "[With wicked problems,] the planner has no right to be wrong."

Not only should planners be part of the modelling and shaping process, but also the potential "consumers" or "victims" of said planning. GMA allows for - almost insists upon - this type of stakeholder participation. (For a discussion of this facilitation principle, see Ritchey, 2017).

7.3 Method & Examples

In his main work on the subject, "Discovery, Invention, Research through the Morphological Approach" (1969, p. 107), Zwicky summarises the five (recursive) steps in the process:

"**First step**. The problem to be solved must be very concisely formulated.

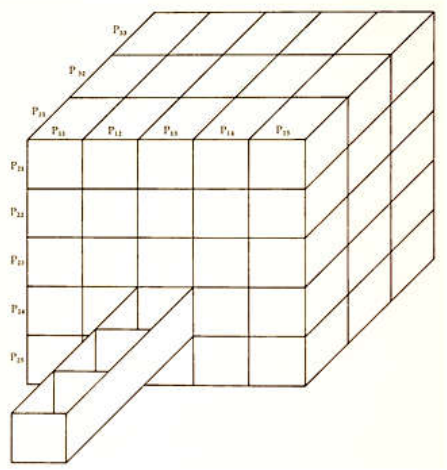
Second step. All of the parameters that might be of importance for the solution of the given problem must be identified and analysed.

Third step. The morphological box or multidimensional matrix, which contains all of the [formal] potential solutions of the given problem, is constructed.

Fourth step. All the solutions contained in the morphological box are closely scrutinized and evaluated with respect to the purposes that are to be achieved.

Fifth step. The optimally suitable solutions are ... selected and are practically applied, provided the necessary means are available. This reduction to practice requires in general a supplemental morphological study."

As concerns the "construction" of the morphological box in the Third step, Zwicky illustrates and represents this in different ways, including those in Figure 7.1. (A third way is shown in Example #1, below.)



Morphological Box of Codes of Conduct

Parameters	Components of the Parameters		
P_1 Conceptual character	P_{11} Dictatorial	P_{12} Mutual Consent	P_{13} Absolutely Objective
P_2 Qualitative character of the standard	P_{21} Body animate, inanimate	P_{22} Phenomenon	P_{23} Concept
P_3 Consequences of deviations from the code	P_{31} None	P_{32} Destructive	P_{33} Constructive

Figure 7.1: Two representation of a "Zwicky box" [220]

Example #1: Designing a new type of telescope

In this first example – from "Morphological Astronomy", Zwicky wants to show how the morphological approach essentially works. After initially formulating the problem to be studied (which sometimes is a formidable task in itself), one begins by identifying and designating all of the possible *quantitative and non-quantitative parameters that are relevant to the problem*. In this case, it would consist of all relevant attributes of telescopes and their performance characteristics. For instance, one significant parameter would be the ratio **r** of the energy entering the telescope aperture to the energy absorbed in the recording instrument. This ratio might be:

A_1: r >1; A_2: r ≅ 1; A_3: r < 1

A second parameter could list all the recording devices available (i.e. photographic plates, ionization chambers, photocells, etc.). A third parameter could describe "the type of interaction of the light with the optical parts of the telescope", e.g. reflection, diffraction, refraction, etc. Such an array of parameters, represented by their respective (discrete) values ranges, is given in the matrix in Figure 7.2. This is the original form that Zwicky gave to a *morphological field.*

If the number of elements in each parameter is respectively n_A, n_B, n_C,... n_X, then the total number *N* of types of possible telescopes is $N = n_A \bullet n_B, \bullet n_C ... \bullet n_X$. In Fig. 7.2, one of these possible combinations of parameters is displayed in ($A_1 + B_2 + C_2 + \ldots + X_3$).

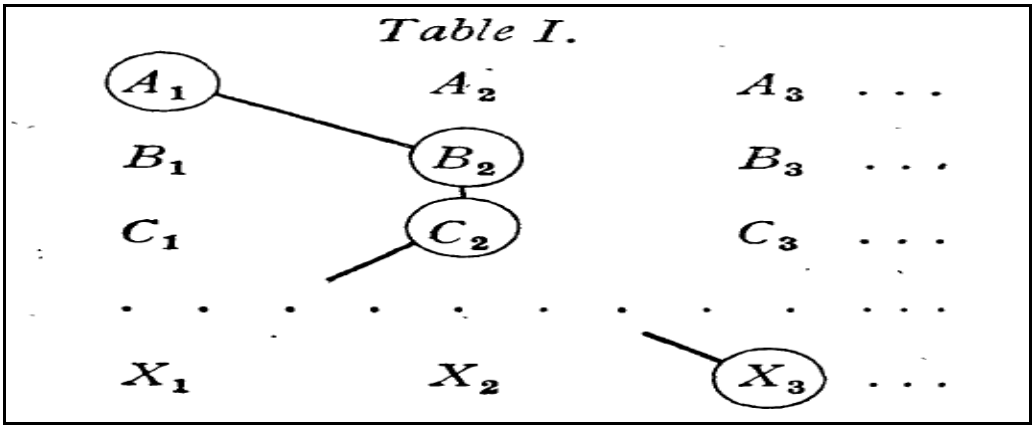


Figure 7.2: Zwicky's original representation of the morphological field for an X-dimensional problem, with one "solution" displayed (circles).[221]

One then carries out a *performance analysis of the telescopes arising from each of all the possible combinations arrived at*. This process represents two strangely superimposed (and what might seem to be mentally contradictory) tasks: on the one hand, of identifying combinations of attributes which are seen to be logically impossible or empirically implausible – and discarding them; and on the other hand, of keeping one's mind open for the discovery of strange and novel combinations that we may not hitherto have imagined. (This exhaustive method of examining each possible "configuration" can be done if the model is small enough. Note that as few as 6 parameters, each containing 4 attributes, will result in 4096 possible combinations.)

Example 2: The general problem of astronomy

Zwicky then goes on to outline the more ambitions challenge of showing how the morphological method might be applied to "the general problem of astronomy", which would include the following areas:

(a) Observation of celestial phenomena
(b) Experimentation with celestial phenomena
(c) Theoretical integration
(d) Use of the knowledge gained in construction
(e) Dissemination of the knowledge and its bearing on all activities of Man.

Now these "areas" can, in turn, be studied as (initially) separate small morphological models. He thus breaks down (a) "Observation of celestial phenomena" into the following parameters:

(α) The instruments to be used for observation
(β) The location of these instruments
(γ) The alternative of manned and unmanned instruments
(δ) The objects of observation – divided into:
  (δ_1) the contents of the universe and their quantitative and qualitative nature, and
  (δ_2) the physical laws governing the interactions of the celestial bodies & general fields.

Although Zwicky does not go on to formulate a full morphological field for these parameters, it would look something like Figure 7.3, rendered in contemporary GMA field-format.

The instruments to be used for observation	The location of instruments	The alternative of manned and unmanned instruments	The objects of observation
Telescopes for different wave-lengths: E.g. gamma ray to radio telescopes	Earth-bound observatories	Manned re-entry recovery	The contents of the universe and their quantitative and qualitative nature
Photo-electric telescope (e.g. CCD)	High flying aircraft	Unmanned re-entry recovery	The physical laws governing the interactions of the celestial bodies and the general fields
Neutrino detectors	Balloons	Telemetrics back to earth	
Gravitational wave detectors	Non-orbiting rockets		
Dark matter detectors	Earth-orbiting satellites		
	Moon-based		
	Interplanetary		
	Interstellar		

Figure 7.3: Hypothetical morphological field for Zwicky's outline for "Observation of celestial phenomena"

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Example 3: A Morphology of Propulsive Power

The final example is taken from one of Zwicky's most well known works in the area of engineering design: "Morphology of Propulsive Power" (Zwicky, 1962). Here he categorised and exemplified 576 theoretically possible modes of propulsion systems using six parameters (Figure 7.4).

Character of chemical reactions	Mechanical character of propulsion system	Method of thrust augmentation	Physical state of propellants	Operating mode of propulsive power plant	Reactivity or Reaction Speed of the Propellants
Self-contained - carries all chemicals necessary for activation and operation.	No motion	No thrust augmentation.	Gaseous state.	Continuous operation.	Propellants are self-igniting.
If air-propelled, carries only fuel and uses atmospheric oxygen.	Translatory motion.	Internal thrust augmentation.	Liquid state.	Intermittent (pulsating) operation.	Artificial ignition is necessary.
If propelled through or over water, uses water as propellant reacting with an on-board water-reactive chemical.	Rotary motion.	External thrust augmentation.	Solid state.		
If propelled through or over the earth may use earth as propellant reacting with an on-board earth-reactive chemical.	Oscillatory motion.				

Figure 7.4: Zwicky's propulsive power morphology containing six dimensions (parameters) and 576 (4x4x3x3x2x2) formal configurations – one displayed.

While *Morphological Astronomy* and *Morphology of Propulsive Power* remain among his most well known works, Zwicky made scores of morphological studies from the late 1940's until his retirement in 1969. Many of these still languish, unpublished, in the Zwicky Archives in his home town of Glarus in Switzerland. However, we can get an idea of the diversity of his work in some of the titles he mentions in other works. These include "Morphology of Justice in the Space Age"; "Morphology of Multi-Language Teaching"; and "Morphology of Codes of Conduct in Law and the Administration of Justice".[222]

Now, the examples presented here may seem to be relatively "simple", even as compared to those of Ramon Llull from the 13th century. However, Zwicky was not interested in theological or metaphysical exercises, but practical sociotechnical, legal and ethical problems that could actually be tackled with combinatorial modelling. This means that they need to be small enough to be combinatorially treatable "by hand". Models involving as few as 6–7 parameters with domains 5–6 elements become totally unmanageable without computer support.

However, computer applications in such areas as operations research and engineering design were only beginning to be developed in the 1960s and 70's. Furthermore, they initially relied on behemoth mainframes which were hardly "user-friendly". Flexible, integrated graphical interfaces designed for non-quantified combinatoric modelling were practically non-existent. The major breakthrough in CAD and related applications had to wait until the 1980's with the development of PCs and Workstations. If inexpensive digital computers with user friendly (graphic) interfaces had been available during the 1950s and 60s, Zwicky would most likely have been interested in employing these for the development of larger, more complex combinatorial models.

From the 1960s to the early 1990's, on the basis of Zwicky's work, several researchers in Europe and the U.S.A. began to explicitly employ "the morphological approach" in Operations Research and Management Science (OR/MS), in engineering and product design, and especially for futures projections and technological forecasting.[223] However, these earlier studies were carried out by hand or with only rudimentary computer support (e.g. early digital spread sheets), which is highly time-consuming, prone to errors, and severely limits the number and range of parameters that can be treated.

Then in the early 1990's, dedicated computer support for "General Morphological Analysis" (GMA) was developed at the Institute for Technology Foresight and Assessment in Stockholm.[224] This has made it possible to create large, interactive, non-quantified (if-then) inference models, which significantly extended the method's functionality and areas of application.[225]

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9. Notes

[1] For background see e.g. Kadison & Kromann (1996): "Historical Foreword"

[2] Whitehead, 1906:4.

[3] I have chosen to cite this later article because it is expressed more succinctly. In APG (1906) he writes: "Geometry, in the widest sense in which it is used by modern mathematicians, is a department of what in a certain sense may be called the general science of classification." (p. 4.) … "In Projective Geometry the subject viewed simply as a study of classification has great interest. Thus in the foundations of the subject [of projective geometry] this conception is emphasized, while the introduction of 'order ' is deferred. (p. 6).

In his last work, "Essays in Science and Philosophy" (1948), Whitehead writes:

"*Definition of Abstract Geometry*. – Existent space is the subject matter of only one of the applications of the modern science of abstract geometry, viewed as a branch of pure mathematics. Geometry has been defined as "the study of series of two or more dimensions." It has also been defined as "the science of cross classification." These definitions are founded upon the actual practice of mathematicians in respect to their use of the term "Geometry." Either of them brings out the fact that geometry is not a science with a determinate subject matter. It is concerned with any subject matter to which the formal axioms may apply. Geometry is not peculiar in this respect. All branches of pure mathematics deal merely with types of relations. Thus the fundamental ideas of geometry (e.g., those of points and of straight lines) are not ideas of determinate entities, but of any entities for which the axioms are true. And a set of formal geometrical axioms cannot in themselves be true or false, since they are not determinate propositions, in that they do not refer to a determinate subject matter. The axioms are propositional functions." (p. 178.)

[4] Written for an article on "Geometry" for Encyclopaedia Britannica (Emphasis added). He goes on to write here:

"… these formal geometrical axioms cannot in themselves be true or false, since they are not determinate propositions, in that they do not refer to a determinate subject matter." (Ibid.)

Earlier, in **APG** (1906, p.2) he writes:

> "… the points mentioned in the axioms are not a special determinate class of entities; but they are in fact *any entities whatever*, which happen to be inter-related in such a manner, that the axioms are true when they are considered as referring to those entities and their inter-relations. Accordingly – since the class of points is undetermined – the axioms are not propositions at all: they are propositional functions." (Emphasis added)

[5] See Henderson (2018) for an intriguing account of how the introduction of non-Euclidian geometry and operative symbolism (and specifically Leibniz's differential and integral calculus) challenged the Locke-Newton program and had both political and cultural ramifications for Victorian society.

[6] This was a continuation of the earlier work of Karl Gauss, Bernhard Riemann and James Maxwell. In this context, Whitehead was specifically influenced by two important figures who had carried this tradition forward: the German mathematician Felix Klein (who was a frequent visitor to Cambridge) and the British non-Newtonian physicist William Clifford (who produced the first English translation of Bernhard Riemann's famous Habilitationsvortrag "On the Hypothesis which lie at the Foundations of Geometry"). Note that Whitehead's concept of projective geometry built on Maxwell's "geometry of position", a modern version of Leibniz *analysis situs*.

> "Maxwell, drawing upon Gauss, had used the term 'geometry of position' in 1871 for the projective geometry of Lazare Carnot and Michel Chasles. Gauss had been inspired by Leibniz's remarks on the need to formulate geometric algorithms to express geometric location (analysis situs)." Dawson (2008): 17, Note 57, in reference to Harman (1998): 154.

For a more detailed account of Whitehead's relationship with Klein and Clifford, see Dawson (2008).

[7] I.e. the move from rhetorical and abbreviate mathematical representation to generative, embedded symbolic representation. See e.g. Serfati 2010, 2020 for detailed discussions.

[8] At this point, the details of Whitehead's actual axioms are not important. See §6 for the three relevant axioms expressed by Leibniz.

[9] Note the association between this and modern mathematical "model theory", which was being

developed during the first half of the 20th century. Indeed, "model theory" emphasizes the complementary relationship between syntax and semantics.

> "Model theory is the study of formal languages (syntax) and meaningful interpretations of such a language (semantics). The interplay between the syntactic and semantic perspective is a fundamental aspect of the subject." (Kelly (2015):1.)

See Benis-Sinaceur (1989/2023) for an account of the relationship between Leibniz, Kurt Gödel, *ars inveniendi* and the development of model theory.

[10] As Serfati (2010:120) puts it: "After the introduction of the symbolic writing system, nothing in mathematics was anymore like before. The outcome was, strictly speaking, a (symbolic) revolution, one of the major components of the scientific revolution."

[11] Whitehead 1906; 1911; 1933; 1947. The nominal definition of *cross-classification* is that of placing items into classes based on the features of two or more variables. This is technically correct, but conceptually limited. Whitehead is using the term in the wider epistemological sense of multivariate cross-order mapping and heuristic symbolic modelling.

[12] Note that we are using the expression *conceptual modelling* in the very general sense of any type of modelling with "concepts" rather than with quantities. The term has also been employed in a more restricted sense in e.g. operational research to denote "causal-loop diagrams" and "activity diagrams". Unfortunately, current notions of "conceptual modelling" are primarily associated with computer-based modelling, which often totally disregard its 2000+ year-old-history as a method for creating compound concepts and generating new knowledge. For instance, in the recent anthology "The Evolution of Conceptual Modeling - From a Historical Perspective towards the Future of Conceptual Modeling", it is stated in the Preface:

> "Conceptual modeling is a rather recent approach to creating knowledge. It has emerged in response to the computer revolution which started in the middle of the twentieth century." (Kaschek & Delcambre, 2011: IX.)

[13] This concerns *inter alia* so-called facet-analysis as applied in contemporary, non-pre-categorized "tagging" systems used in digital search engines. As noted by Hjørland (2013: Abstract), facet-analysis – a cross-classificatory system – was developed in the early 20th as a complementary knowledge organization approach within Library and Information Science. For overviews and further details see e.g. de Grolier (1962); Dahlberg (1976); van den Heuvel (2011).

[14] See the influential work of Dunnell (1971).

[15] ISO (1087:2000) terminologies concerning "types of concept relations" makes the basic distinction between *hierarchical* (or super- and sub-ordinate) and *associative* (or co-ordinate) relationships; and four "general concept relations": *identity*, *inclusion*, *overlapping* and *disjunction*. (see Nuopponen, 2014). See §2 for a derivation of these "operators".

[16] Whitehead (1933:137f). Whitehead's reference to (strict) linear hierarchal classification as "Aristotle's method" is somewhat misleading. Aristotle – like Plato (see § 4, infra) – also clearly recognized the usefulness of cross-ordering concepts (e.g. in "The Categories"). However, it is nonetheless the case that 1) Aristotle considered strict hierarchal classification and its use in deductive inference or "demonstration" as the true "scientific" method of classification, and 2) that this became the historical standard or "tradition" in classification theory. I understand Whitehead as referring to this "tradition". Cf. Hjørland (2017):

> "A now well-established division between two theories of concepts and classification is therefore classical or Aristotelian classification on the one side and *prototype theory* (or

> polythetic classification) on the other." (p. 103) …. "A monothetic class is defined in terms of characteristics that are both necessary and sufficient in order to identify members of that class. This way of defining a class is also termed the *Aristotelian definition of a class*." (p. 114)

[17] On a note from an anonymous reviewer, and on the advice of a Swedish classical scholar, I have abstained from using *diacritics* (accents, umlauts, etc) associated with isolated classical Greek terms, as they are generally superfluous in the context of the present work. However, I have retained them when directly citing Greek text segments. For Latin alphabet transliterations I have used the *Lexilogos* multilingual keyboard.

[18] This is Debuiche's (2013: 359f) description of Leibniz's similar work in developing a *Geometrica characteristica* (symbolic geometry) and a logical calculus, which has been called "The Leibniz Project". Cf. Loemker (1969): 371–382. See §6, *infra*.

> "During his whole life, Leibniz attempted to elaborate a new kind of geometry devoted to relations and not to magnitudes, based on space and situation, independent of shapes and quantities, and endowed with a symbolic calculus. Such a "geometric characteristic" shares some elements with the perspective geometry: they both are geometries of situational relations, founded in a transformation preserving some invariants, using infinity, and *constituting a general method of knowledge*." [Debuiche, 2013. Abstract.)

[19] Whitehead 1998: v–viii. "Such algebras … are worthy of a comparative study, for the sake of the light thereby thrown on the *general theory of symbolic reasoning…* (Emphasis added.)

[20] "A Study in the Calculus of Real Addition", in Leibniz (1966: 142).

[21] Leibniz presented the first manifest of this notion in a paper published in Acta Eruditorum (1684) titled "Meditationes de Cognitione, Veritate et Ideis [Reflectons on Knowledge, Truth and Ideas]. What was first accomplished quantitatively in mathematics (Viete & Descartes) Leibniz wanted to further develop, on a purely conceptual bases, with a calculus (or algebra) of concepts.

[22] Swoyer (1991).

[23] Kanamori (2009). 474.

[24] Krämer (1997, 2003, 2016); cf. Esquisabel (2012).

Krämer also calls it "operative writing" and describes it as:

> "... inspired by three attributes: (1) Symbolic language can be used as a technique for problem solving. (2) The rules of manipulating symbols can be independent of their interpretation. (3) Symbols do not only depict, but constitute knowledge." (Krämer 2016, Abstract.)

[25] "Grassmann looked back to Leibniz and claimed that that the geometric algebra developed in his groundbreaking 1844 *Ausdehnungslehre* was the natural development of Leibniz's ideas on the 'geometric characteristic', announced in A letter to Huygens in 1679. Grassmann's prize essay of 1847 was written in response to a challenge to complete Leibniz's project for the characteristic." (Banks 2013: 21).

Writing about his own work in the third person, Grassmann states: "[Leibniz presents] yet another remarkable point where he quite clearly expresses the applicability of this [geometrical] analysis to objects that are *not of a spatial nature*, but adds that it is not possible to give a clear concept of this in a few words. Now, in fact, as is demonstrated throughout Grassmann's *Ausdehnungslehre*, all concepts and laws of the new analysis can be developed *completely independently of spatial*

intuitions, since they can be tied to the abstract concept of a continuous transformation; and, once one has grasped this idea of a pure, conceptually interpreted continuous transformation, it is easy to see that the laws developed in this essay are also capable of this interpretation, stripped of spatial intuitions. Grassmann 1995: 384; cited in Banks (2013: 21).

[26] See e.g. Magnani, *et. al.* (1999). Thanks to Lorenzo Magnani *et. al.* there is a whole library of books and articles on this subject.

[27] "Deliberative arts" refers to practices that employ methods to facilitate understanding, critical thinking, and collaborative decision-making, often in complex or contested social or environmental contexts. These arts emphasize inclusive communication, exploring diverse perspectives, and generating new understandings rather than just finding a single right answer. (Adapted from an AI-generated overview from Google.)

[28] For a detailed review see e.g. Feest & Sturm (2011); Sturm (2011); Nickles (2017); Vagelli (2019). Note that the original French concept has been variously translated into English a "Historical Epistemology", "History of Epistemology" and "Epistemological History". There has been some discussion as to the possible differences in the meanings of these terms, which will not be addressed here. For such possible distinctions see e.g. Gattinar, 2023; Gingras, 2010; Tiles, 1987.

[29] Atack (2016). Cf. Rood et. al. (2020); Vagelli (2019).

[30] Gambarotto (2019:150). (Emphasis added)

[31] For a detailed account see Mauján (2020).

[32] E.g. Wilhelm Ostwald noted that farsighted educators "... have repeatedly been forced to point out a deficiency which too often attaches to the present scientific education of our younger talent. It is the absence of the historical sense and the want of familiarity with the great researches upon which the edifice of science rests." Cited in Cajori (1899): 278. Cf. Ostwald (1910).

Cf. Ernst Mach: "They that know the entire course of the development of science, will, as a matter of course, judge more freely and more correctly of the significance of any present scientific movement than they, who, limited in their views to the age in which their own lives have been spent, contemplate merely the momentary trend that the course of intellectual events takes at the present moment" (Mach 1960: 8f, cited in Maienschein. (2000): 343.)

[33] The opening paragraph of Leibniz's "Historia et Origo Calculi Differentialis" (1714). Also: "Nothing is more important than to see the *sources of invention* which are, in my opinion, more interesting than the inventions themselves". Cited in Polya 1945, p. 123.

[34] See Ritchey (2018) for a more detailed discussion.

[35] Note that the terms "modelling theory" and "modelling theoretical" used here are not synonymous with (mathematical) "model theory", although there is, of course, *some* common ground. See e.g. Ritchey (2012).

[36] The *Entity-Relationship Model*-concept was formalized and popularized by Peter Chen (1976) as a method of conceptual modelling for digital database development. However, it has a much longer history. Methodologically it is based on the so-called *structuralist approach* in which a system, or a model:

Cf. Grossmann (1990: 251.):

> A structure is [a representation of] a complex entity which consists of parts in relations to each other. It differs from a set because of the relations which it contains. … Two sets are the same if and only if their members are the same. Two structures are identical, by

> contrast, if and only if (a) their nonrelational parts are the same, (b) their relations are the same, and (c) corresponding parts stand in corresponding relations."

[37] The term parameter is being used here in its broader "systems science" sense, as being one of a set of factors that defines a system and determines its behaviour, and which can be varied in an experiment – including a *Gedankenexperiment*.

[38] See e.g. Lawvere & Schanuel (2009) on "Conceptual Mathematics".

[39] "An instance of the *Constraint Satisfaction Problem (CSP)* is defined by a set of variables, a set of possible values (or *domain*) for each variable, and a set of constraining conditions (*constraints*) involving one or more of the variables. The *Constraint Satisfaction Problem* is to decide for a given CSP instance whether all variables can be assigned values from their respective domains such that all constraints are simultaneously satisfied." (ScienceDirect.com: (https://www.sciencedirect.com/topics/computer-science/constraint-satisfaction-problems#)

[40] Not to be confused with "multi-modal modeling" in AI which is about integrating various "data types" such as text, images and audio.

[41] De Beer (2023:7)

[42] For the *iconic* aspect, see the thought-provoking works of Jamin Pelkey (2016; 2017; 2021).

[43] Ong (1982:34): "In a primary oral culture, to solve effectively the problem of retaining and retrieving carefully articulated thought, you have to do your thinking in mnemonic patterns, shaped for ready oral recurrence. Your thought must come into being in heavily rhythmic, balanced patterns, in repetitions or antitheses, in alliterations and assonances, in epithetic and other formulary expressions. … Mnemonic needs determine even syntax." Cf. Havelock (1963:87ff, 294ff). See also Kelly (2015) especially Chapter 3: Primary Orality and Oral Mnemonic Technologies. Such "preservation" does not imply a "fixed" structure. Oral tradition was never fixed but was in continual flux. Indeed, it became relatively "fixed" when written down.

[44] It is found in the Sumero-Akkadian and Ugaritic texts as early as the third millennium B.C. (Breck, 1994; cf. McCoy, 2003). It is commonly preserved in all of the great literature which has oral foundations: e.g. Bible, Quran, Beowulf, Homer (Iliad & Odyssey) and Ancient Greek plays.

[45] Indeed, forming and interpreting conjoined concepts ("conceptual integration") is ubiquitous to human cognition:

> "A striking characteristic of human cognition is our capacity to produce and comprehend conceptual combinations that merge previously separate concepts into units that express new thoughts and stimulate new ideas." (Wilkenfeld & Ward (2001): 21)

> "… chiasmus proves to be a remarkably compact and powerful engine for organizing, inflecting and generating ideas." (Hariman (2014): 49)

See also Hampton (1997a); Fauconnier & Turner (1998; 2002).

[46] Merleau-Ponty (1968), Chapter 4 (Heading) and p. 138.

[47] The so-called *ring structure* is a cross-symmetrical or reversed pattern of recurring phrases or themes that can structure a complex passage, narrative or whole book. Its depicted form is: A-B-C-D-E-F::F-E-D-C-B-A. See e.g. de Beer (2023). The *argyle pattern* "… entails replicating 'X'-patterns, dovetailing and crammed together. There is a shared communal space between the symbols" (Pelkey, 2017:193–225).

[48] The motif of Dumas' "The Three Musketeers". However, it comes from an old Latinate phrase *Unus pro omnibus, omnes pro uno.* It was used by Shakespeare in his poem *The Rape of Lucre* (1594). It is also the "unofficial motto" of Switzerland.

[49] This label was suggested by Fahnestock (1999):150. Cf. Sullivan (2017).

[50] The version attributed to Socrates by Plutarch in *"How a Young Man Ought to Hear Poems"* reads: "Bad men live that they may eat and drink, whereas good men eat and drink that they may live". Another version is: "The rest of the world lives to eat, while I eat to live" – attributed to Socrates by Diogenes Laertius: *Lives of the Philosophers* bk. 2, sect. 34. One might suspect that the idea of this antimetabole predates the Greeks.

[51] From Fahnestock (1999):151.

[52] Koestler (1964):120.

[53] It was one of the principle inspirations for Fauconnier & Turner's (1996, 1998, 2002) popular work concerning conceptual integration or "blending theory".

[54] Cf. Pekley (2016):32.

[55] Here, life and death are treated as contraries, which might bother some people. We may either allow for this in rhetorical structure, or allow for one of the bisociate pairs (but not both) to be contradictory.

[56] There are many variants. See e.g. https://quoteinvestigator.com/2013/06/17/good-original/

[57] Cf. Sullivan (2017).

[58] Note here that chiastic **AB::¬B¬A** is fundamentally different from, and far more complex, than the negation appearing in a corrective antimetabole: **AB::¬(BA)**.

[59] See e.g. Greimas & Courtés (1982) for a thorough account.

[60] Based on work by Peirce and Saussure. Cf. Petitot (2009); Vermeiren (2019); Schroede (2022).

[61] Greimas & Courtés (1982):311. Rossolatos (2012): "The aim of the square is not to portray semic relationships in a universally binding and logical manner, but … [in a way] which a culture (in a sociolectal semantic universe) or an individual (in an idiolectal semantic universe) perceives the relations among certain entities." (p. 11.)

[62] Greimas regarded the "structural conception of language as simultaneous *contrasts* and *combinations* …" where narrative discourse is "… neither pure contiguity nor [strict] logical implication". (Greimas (1983): 244, cited in Schleifer (1987:4)). As concerns chiastic structure:

> "…the elemental structure of signification is 'present in a double aspect'; it is a structure of (1) contrasts and (2) combinations" which is "susceptible to cross-reference." (Schleifer (1987):10)

[63] Adapted from Pelkey (2017a):6.

[64] For an overview of the "Square of Opposition" and its historical development see e.g. Westerstahl (2012); Parsons (2021). The Euler diagram was taken from Lemanski (2016). Euler's text concerning this figure: "These circles, or rather these spaces, for it is of no importance of what figure they are of, are extremely commodious for facilitating our reflections on this subject, and for unfolding all the boasted mysteries of logic, which that art finds it so difficult to explain; whereas, by means of these signs, the whole is rendered sensible to the eye." [Euler 1833: 340. Cited in

Lemanski (2016):106.]

[65] "The square of Opposition finds its roots in the dialectical discussions on negation, falsity and non-being in Plato's *Sophist*, and so the square finally stems from Plato's solution to the problem of how non-being is, if one does neither accept that everything is true nor that everything is false." (Correia, 2012:51.)

[66] For an account of Apuleius see Londey & Johanson (1984). For a detailed account of the whole development from Apuleius, Ammonius and Boethius, see Lemaire (2017).

[67] Garson (2024).

[68] Parrochia & Neuville (2013:125). (Emphasis added.)

[69] Serres (1972):13. Cited in Sutherland (2021).

[70] And seems also to be present in many animal species. See e.g. Turner et. al. (2021)

[71] See Tennis (2015) for a detailed discussion. Cf. Bawden & Robinson (2022) who denote a similar (but not identical) three level structure: 1) "the level of day-to-day routine practice", 2) "the level of the delineation of the discipline and its knowledge base" and 3) "the foundation, and necessarily abstract, level." (p. 74.)

[72] For further accounts of this issue see e.g. Dunnell (1971:84f), Marradi (1990:§2.2).

[73] Ellerman (2013:2): "Partition logic is at the same mathematical level as subset logic since models for both are constructed from (partitions on or subsets of) arbitrary unstructured sets with no ordering relations, compatibility or accessibility relations, or topologies on the sets."

[74] Cf. Jardine & Sibson (1968):179; Gibson, (1908):44.

[75] Greene's work has been reported – and expanded – by Brumbaugh (1961, 1965), the latter of which has been my main source.

[76] Fig. 2.3a: On Augustine's "On Christian Doctrine" (1212) (British Library). Fig. 2.3b: From a 10th century manuscript of Boëthius' *De institutione arithmetica* (Einsiedeln, Stiftsbibliothek). From Marchese (2013).

[77] Johansen (2014): 103.

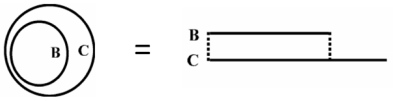
[78] In classical Greek, one of the terms used for "definition" (ορισμός = *horismos*) is based on a spatial metaphor: i.e. "marking off a boundary". Its Latin translation *dēfīnītiō* and its English equivalent *definition* retain this metaphor in the term *fīniō*: to bound, make finite or set a limit.

[79] Kwasnik (1992): 63 (Emphasis added.)

[80] The principle involved in *faceted classification* is that an object is assigned (or defined by) multiple intersecting (or coordinate) categories which are orthogonal, symmetric and non-hierarchal. These are used as "tags" by which the object can be identified in a search. Interestingly, this is also called an *analytico-synthetic* scheme, which highlights the fact that this is a typological/morphological *modelling space*. Although faceted classification can also employ taxonomic structure *within* one or more of its "facets", its primary, "global" structure is typological. Cf. Levy (2011); Frické (2011). For a history of combinatory classification see Glymour, et. al. (1995); Schulte-Albert (1974, 1979).

[81] Note that the use of "Euler diagrams" did not start with Euler, who made no first claim to it. First of all, such diagrams are implicate in Plato's spatial metaphors concerning class inclusion and

concept blending (§4); Ramon Llull is credited as being one of the earliest to explicitly use such diagrams in his texts (§5); and Leibniz made the first systematic study of them, showing how one could *inter alia* represent all valid syllogisms with them. Leibniz also invented an equivalent system base on line diagrams. Cf. Bennett (2015): 108.



[82] *Class disjunction* is what is called *differentia* in *species/differentia* schemes of syllogism. "... the power of Euler diagrams and semantic networks, both representing only two types of logical relation, *entailment* and *contrast*/*opposition*, has been demonstrated through examples of the representation of significant conceptual structures such as taxonomies..." Gaines (2010):30f.

[83] Cf. Gaines (2010). Pavlinov (2011): Abstract: "Rational taxonomy is based on the assumption that the key characteristic of rationality is deductive inference of certain partial judgments about reality under study from other judgments taken as more general and *a priori* true." Doyle & Patil (1991): 261f: "Taxonomic reasoning …is essentially equivalent to *logical entailment*, and so capable of supporting most forms of *logical inference*." (Emphasis added).

[84] We can distinguish between the "synchronic constitution" of a proper taxonomic hierarchy as representing a sequential synthetic-deductive structure, and its "diachronic development" as a form of knowledge generation. See e.g. Andersen (2002):109.

[85] McKinney (1969): Abstract.

[86] As a set theoretical operation this is called a *cross product* or *cross join*; in mathematics a *Cartesian product*. For *concept* analysis and synthesis, Leibniz called this "real addition" and gave it the conjunctive operative symbol $\oplus$ (see §6, *infra*). If differs from a Cartesian product in that its conjunctive elements are combinatory and need not be ordered. See Swoyer (1994) for a detailed account.

[87] "When several *fundamenta* are jointly considered, a typology is produced. This may be done through either intensional or extensional classification. The underlying category space may be 'reduced' or reconstructed through 'substruction'". (Marradi (1990), Summary: 129.)

Note that in statistical analysis this typological classification structure is called a *contingency table*, i.e. a tabular representation of categorical data showing e.g. frequencies for particular combinations of values of discrete random variables. When two variables are cross-classified it is called a *two-way contingency table*; with three variables, a *three-way contingency table*; etc. See e.g. Agresti (2007), Chapter 2. *Contingency tables* are a specific form of typology.

[88] Although there are some similarities, what is meant by "domain extension" here – in the context of typological analysis – is not synonymous with its use in mathematics, where it is the "…*extension* by one or more *sets of objects* in order to obtain a simplification and unification of the relations between the objects of a domain." [Majer (1998):232. Emphasis added].

[89] For the record: Note that in this example, "house-boat" and "boat-house" are not (linguistically) equivalent conceptual combinations. The former, called an "intersective combination", is the logical conjunction denoting something that is *both* a house *and* a boat; whereas the latter, called a "modifier-head combination", is a sub-ordinate category denoting a specific "type of house" – like dog-house, warehouse and lighthouse – but not *also* a boat. Cf. Hampton (1997b) for a more detailed discussion of linguistic conceptual combination.

[90] Cf. Rivet, et. al. (2016).

[91] Körner (1976): 6.

[92] Beyond these three (ontology, epistemology and ethics) even psychology and aesthetics may be added. However, I agree with Havelock (1963:30) that the "theory" is primarily epistemological: "… it seeks to define the character of that knowledge which we would call universal…".

[93] "It is a gift of the gods to men, or so it seems to me, hurled down from heaven by some Prometheus [alluding to the Pythagoreans] along with a most dazzling fire." (Philebus, 16c). Indeed, Plato's debt to the Pythagoreans goes far deeper than this, to the extent that he is considered to be an ancient Neo-Pythagorean. For further discussion of Plato's relationship with the Pythagoreans see e.g. Huffman, 2001: 68–71; Kahn, 2001.

[94] One of the most historically influential examples, accredited to Hippocrates (460–370 BC), is the bisociate chiastic model involving the two dimensions *hot-cold* and *dry-wet*, which produce the four states of nature: Earth, Air, Water and Fire.

[95] This does not mean that Plato conceived of "models" in exactly the same way that we do today. *Paradeigma*: from *para-* (beside, near) + *deiknumi* (to show), is variously translated as "to compare" or "to set up an exemplar, type or model". Thus in this context, what Plato means by a "model", particularly in the Sophist and Statesman, is an existing comparative analogical *structure* or *function* which can be employed as a *surrogate* in order to aid in the investigation of another (usually more complex) structural/functional complex. (For further discussion see e.g. Gill (2006); Frede (2020).)

[96] Plato sees this method of concept classification as not only in the service of science, but that it is actually what makes it possible for us to think and reason *at all*. In the Sophist the "Visitor" talks about "... the *weaving together* of forms is what makes speech and reasoning possible for us." (Sophist 260a.) And in the Phaedrus (266b), Socrates states: "I am myself a lover of these divisions and collections, so that I may *be able to think and to speak*." It is hardly anachronistic to compare this to Fauconnier and Turner's modern account: "Mental spaces are small conceptual packets constructed *as we think and talk*…". (Fauconnier & Turner (1996): 113.).

[97] "'Dialectic' has acquired multiple, often conflicting, meanings in the history of western philosophy. It has been identified with rhetoric, sophistry, Socratic 'cross-examination', Platonic ascent from the sensible to the spiritual, late-Platonic definition by division into genera and species, Aristotelian sifting of opinions *pro* and *con*, Kantian 'transcendental' illusions of the understanding, Marxian socioeconomic stages through capitalism to socialism, etc." (Kainz (1996):1; cited in Rescher (2007):119.) Kainz might well have added Hegel and Fichte to the soup. For a further critique of the 19th and 20th century misuse of "dialectic" see e.g. Hoffman (2005b).

[98] Evans (1977): Preface.

[99] Rescher (2007):139.

[100] Kaldis (2008):68. "[T]he role of the Dialectic itself as a more basic ... method of argument or rigorous philosophical intuition that transcends and supports ... the separate methodologies of different scientific fields." Cf. Stekeler-Weithofer (1992).

[101] "Vertical loops*"* is taken from cybernetics. "Stratified interaction*"* is used in several disciplinary contexts including sociology, information retrieval (IR) and medical science (cf. Saracevic (1997). Likewise "recursive-hierarchal systems" (see e.g. Neuman (2004)). For one of the most thorough discussions available concerning the notion of "dialectic", see Rescher (2007).

[102] "Plato describes, in an intuitive way, certain relationships among the forms. These relationships determine, in turn, certain logical relationships in language… The task of logical theory is to provide a formal representation for the logical interconnections thus exhibited." (van Fraassen,

1969:492).

[103] Jowett (1892) refers to Platonic "cross-division" in the Introduction to his translation of Sophist. Add to this Cornford (1935), Shorey (1924), Cherniss (1945). All of these scholars rightly maintain that Platonic *diairesis* also involved symmetric (co-ordinate-conjunctive) relationships. For further discussion see e.g. Sayre (1969) (The Mingling of the Forms):182–203.

In his book "Plato's Mathematical Imagination", Robert Brumbaugh emphasizes Plato's use of "spatial orientation to differentiate kinds of relation" (Brumbaugh, 1954:20). He calls these *verbal matrices* which "… define a term by locating it in relation to a set of other systematically ordered terms" (p. 72). In "Logical and Mathematical Symbolism in the Platonic Scholia" (Brumbaugh 1961:45) he points out Plato's particular use of "*cross-classification* in *combination matrices*". Other terms have used by different scholars, including lenses, meshes, quilts, networks and lattices (see Pasqualoni, 2016).

[104] Among these are Gill (2012; 2019), Ionesco (2013), Pasqualoni (2016), Smith (2018, 2019, 2020), Frede (2020),Pavani (2023), Liu (2021, 2024). Frede (2020).

[105] "I argue that the Eleatic Stranger's account of what the dialectician "adequately views" at Sophist 253d1-e3 involves both division and the communion of ontological kinds – and not just one or the other as has usually been argued." Smith (2020):Abstract.

[106] Benjamin Jowett – one of its modern translators and interpreters – states that the Sophist contains the seeds of just about everything that scientific methodology requires. It is regarded as the basis for Aristotle's development of syllogistic logic and well as the precursor to "modern" compositional analysis and synthesis. Martin Heidegger gave a famous lecture series on it (1924–25) that influenced a whole generation of epistemologists and historians of science. (Heidegger, 1997)

[107] As far as I can ascertain, the term "dialectician passage" was first used by Colin Smith (2020).

[108] Lloyd (1952):110, referring to Stenzel (1931).

[109] "Being", "motion"," rest", "sameness" and "difference". The discussion of how these great kinds relate to one another is a fascinating – if not to say mindboggling – experience in itself, but is not necessary for the present discussion.

[110] Plato is famously adverse – for the most part – to using fixed technical terms, and in the Sophist we see him using several terms – variously translated as kinds, forms, classes, genera, ideas, wholes – more or less synonymously. This, of course, has added to the enigmatic nature of this passage.

[111] Translation by H. N. Fowler in Plato (1921).

[112] E.g. the remarkable case of Martin Heidegger (1997:365) in his famous Lectures Series at the University of Marburg (1924-25). In discussing the meaning of *Sophist* 253d5-e2 bluntly states:

> "I confess that I do not genuinely understand anything of this passage and that the individual propositions have in no way become clear to me, even after long study."

[113] Moravcsik (1992):217.

[114] E. g. Van Fraassen (1969); Sayre (1969); Harte (2002).

[115] Sayre (1969):193 (emphasis added) u/ "The Mingling of the Forms".

[116] Van Fraassen (1969):493. "We allow each Form A to be represented by a region H(A) of the logical space H, and understand "x participates in A" to be true if and only if x's location in H is within the region H(A). Then, that A and B combine is represented by an overlap of H(A) and

H(B). By designating a set E of "occupied" points (locations of real things in the logical space) we can represent the blending of A and B by an overlap of E and H(A) and H(B)."

[117] Translated text compounded from Fowler (Plato, 1921) and White (Plato, 1959). My brackets and emphasis.

[118] "It is easy for the modern reader to forget that Plato was heir to a pedagogical tradition in which the use of mathematical illustration seems to have been a standard device for clarifying discussion. It is equally easy to forget that behind this tradition lay the new discovery of mathematics as a promising instrument of science, and that successes with it in music and astronomy must have led one to expect equally effective applications to politics and medicine." (Brumbaugh, 1954:15.)

[119] For a more detailed discussion see Liu (2021).

[120] Here one might add that the distinction between "legal" and "illegal" democracies – regardless of what Plato may have meant by this – can certainly be considered a relevant modern-day issue.

[121] Liu (2024):60. "I draw the same conclusion as Sayre. We both focus on three pairs but treat them differently. These three pairs are (1) the pair of antecedents 'if the One is–if the One is not', (2) the pair of subjects 'the One–the Others', and (3) the pair of relations 'in relation to the One–in relation to the Others'. By combining (2) the pair of subjects with (3) the pair of relations, I address the fact that these two pairs are cross-combined, thus constructing a 2 x 2 chiasmus; in addition, the two 2x2 chiasmata are incorporated into the framework established by (1) the pair of antecedents" See further pp. 60ff. C.f. Haecker (2014).

[122] *Parm*.136b7- c8 (my emphasis).Translated by Mary Louise Gill and Paul Ryan, In Plato (1997): 401f.

[123] The eight hypotheses presented by Liu (2024:46) are:

"If the One is, what follows:
(I) for the One in relation to the One/itself (137c4–142a8);
(II) for the One in relation to the Others (142b1–157b5);
(III) for the Others in relation to the One (157b6–159b1);
(IV) for the Others in relation to the Others/themselves (159b2–160b4).

If the One is not, what follows:
(V) for the One in relation to the Others (160b5–163b6);
(VI) for the One in relation to the One/itself (163b7–164b4);
(VII) for the Others in relation to the One (164b5–165e1);
(VIII) for the Others in relation to the Others/themselves (165e2–166c5)"

[124] Some trace it back to the Greek rhetorician Hermogenes of Tarsus during the 2nd century AD. See e.g. Wooten (2012); (Paul (2014).

[125] §44: 21–24. Diagram adapted from Simplicius (2003). (Emphasis added).

[126] Cited in MacLennan (2021):83.

[127] Left (1976):32, cited in Johnston (1987). Cf. Yates (1982):9: "[T]he great thirteenth century which saw the development of scholasticism out of the re-discovery of Aristotle."

[128] Seghedin (2007):1.

[129] Sale (2011):26f. The eminent Llull scholar Anthony Bonner puts it another way: "[The] trick was to ensnare his interlocutors with unobjectionable foundations, whose consequences they would then be unable to refute." Bonner (2007):284.

[130] See Rescher (1962) concerning Al-Farabi's promotion of non-Aristotelian inference and the epistemological distinction between *generating ideas* (*inveniendi*) and *demonstrating proofs* (*demonstrandi*).

[131] See Yates (1966):177–179. "Lull was not a Scholastic, he was a Platonist, and in his attempt to base memory on Divine Names, which verge on Platonic Ideas in his conception of them, he is closer to the Renaissance than to the Middle Ages" (p. 176).

[132] Bonner (2007):81 (My emphasis and brackets). See also p. 272.

[133] Beaney (2011):91.

[134] Mugnai (2010):302. Cf. Pombo (2010):1: "... starting on the basis of categories of universal application and operating with a system of symbolic notations and combinatorial diagrams, Lull establishes the principles of a *synthetic and inventive procedure* which, unlike the demonstrative logics of Aristotle, would not be limited to ... truths already known but would make possible ways for the discovery of new truths." Note, however, that Llull's signs are purely abbreviative and do not constitute a true operative symbolic system.

[135] Biggs (1979):113: "In any discussion of the antiquity of combinatorial calculations, the contribution of the ancient Greeks must be assessed. The assessment is remarkably negative. There are very few relevant remarks in the whole of extant Greek literature …".

However, more research has shown that there was indeed ancient Greek works on combinatorics, including those of Chrysippus (279–206 BCE), Hipparchus (190–c. 120 BCE) and Archimedes (c. 287 – c. 212 BCE), but they were not preserved and transmitted. See e.g. Acerbi (2003); Bobzien (2011). Also, Nicomachus (c. 60– c. 120 AD) is regarded as a number theoretical specialist, but this is primarily from the perspective of Pythaean Theosophical Numerology, not technical combinatorics. Cf. Kahn (2001), pp. 110–118. Cobeli & Zaharescu (2013) suggest that Apollonius of Perga (3rd Century BCE) knew of the Arithmetic Triangle, which is a sure mark of combinatorics knowledge.

[136] Wilson & Watkins (Eds.) (2013).

[137] However, there is one area of interest which seems to be universal, and may represent the independent geneses of combinatorial curiosity across many cultures: Language. Both spoken and written language are inherently combinatoric. And this is one area that Plato and the Classical Greek philosophers did treat. In the Sophist, Plato remarks about how different letters of the alphabet can and cannot combine. Xenocrates (396–314 BCE), Plato's close friend and head of the Platonic Academy from c. 339 to 314 BC, made an attempt to calculate the total number of syllables which could be constructed from the letters of the (Greek) alphabet. The result which Xenocrates obtained was, according to Plutarch, 1,002,000,000,000. This possibly "represents the first [European] attempt on record to solve a difficult problem in permutations and combinations." See e.g. Heath, 1981:319. (Cited in Stanley, 2011:105). Another associated area of general interest for combinatorics concern musical and poetic *meter*.

[138] Divakaran (2016):64.

[139] See Oaks (2012); Cf. Heeffer (2012).

"With regard to simple combinatorial principles, the Arabs seem to have acquired the techniques used by the Hindus and applied them in similar ways. An interesting example occurs in the work of al-Halil Ibn-Ahmad (718–791 AD) … [who] considered the *possible arrangements of letters in the formation of syllables*, and his calculations show that he understood the basic formulae for finding numbers of permutations and combinations." (Biggs (1979):117f (Emphasis added))

[140] This last development was especially important:

> "... Ibrahim ibn Sinan developed a pragmatic logic in which he coordinated an *ars inveniendi* with an *ars demonstrandi.* He was followed by al-Sijzi, who wrote an *ars inveniendi* based on Thabit ibn Qurra and his grandson. Then came Ibn al-Haytham, who developed an analytical art, and, later on, al Samaw'al, who examined *analysis* and *synthesis* in algebra. The world of mathematics would have to wait until late in the 17th century, with Leibniz, for any contribution of comparable importance." Rashed (2009):12.

[141] For further details see Djebar, (2013:85); Link. (2010:217).

[142] Gray (2018):297. Ref: Ibn Khaldūn (1958).

[143] Sales (2011) p.32. One might ask if such "letter magic" is especially suited to Arabic *abjad* (syllabic or consonantory) scripts, which omit some or all vowels. This would mean that the output of such a "combinatoric calculator" would produce more potential interpretations of symbol combinations and permutations than with a more constrained "vowel alphabet". Would a vowelless syllabic script have a greater potential for *combinatoric poiesis*?

[144] From Djebbar (2013).

[145] Rossi (2000):29.

[146] Link (2010): 216. "…the disk construction served theoretical functions as an encyclopedia of religious thought, a tool to inspire meditation about its main topics, and a way to *generate new propositions*."

[147] Yates (1982):7.

[148] "[The] Names of God belong very strongly to the Christian tradition; many of them are mentioned by Augustine, and in the *De divinibus nominibus* of Pseudo Dionysius they are listed at length. The names used by Scotus Erigena (c.815–77) and by Ramon Lull are nearly all to be found in the book *On the Divine Names* of Pseudo Dionysius" [i.e. the anonymous author who incorporated the Neoplatonic doctrines of Proclus into Christianity.] (Yates (1966):177)

[149] Cf. MacLeenan (2007).

[150] Yates (1979):13.

[151] Cited in Bonner (2007:125f.

[152] See e.g. Bonet (2011).

[153] See Welch (1990) for a more detailed exposition.

[154] See Baron (1969):115.

[155] Wilson & Watkins (2013):16.

[156] McLean & London (1992).

[157] See "Lullism as an Art of Memory", in Yates (1966).

[158] *Ibid*. p. 175.

[159] Höffe (2003):195f: "One could say that this acknowledgment of Aristotle's philosophy constituted the birth of the faculty of arts … Soon it became the norm everywhere, and in many places remained so until the end of the Middle Ages, that Aristotle was the core, and often the exclusive content, of the study of philosophy." [It was seen as] "... a coherent set of dogmas and therefore as

something that is finished and completed; thus, [this] philosophy is only a small step away from dogmatism, the mortal enemy of living thought."

[160] "Lull was not a *scholastic*, he was a Platonist, and in his attempt to base memory on Divine Names, which verge on Platonic Ideas in his conception of them, he is closer to the Renaissance than to the Middle Ages." Yates (1966):176.

[161] Cf. Bonner (1997):14.

[162] Mugnai (2010): 302. (Emphasis and brackets added.)

[163] Knobloch (2013):125: "Unlike his followers in the 17th century, Llull was not primarily interested in the mathematical problem of enumerating the possible sign combinations. His combinatorial art was designed as an inventive logic that could judge every posed problem."

[164] Bonner (2007):296f. "One could even say that it is an essential feature of the Art, one that makes the Art itself an exemplary structure. It thus can generate not only answers or demonstrations, but even the questions themselves! It is therefore 'inventive' in a much broader sense than the normal medieval meaning derived from the Topics, of 'finding' strategies for dialectical arguments; Llull's suggestions for its use as an exemplary, self-generative structure, makes its possibilities almost limitless. The ability to apply the Art beyond its own borders make it what Umberto Eco has called an opera aperta, one in which the user is invited to continue the work begun on the pages he has before him."

[165] Bonner (2007):290. Cf. Höffe (2003):25: "In opposition to a Scholasticism gone sterile and content with … *ars demonstrandi*, the alternative, an *ars inveniendi*, accompanied the great scientific awakening of the early modern era. From Bacon to Vico, Wolff and Lambert, by way of Descartes and Leibniz, philosophers in general were looking for a new 'tool'."

[166] Sales (2011):29 "Perhaps the most striking of Llull's anticipations was the idea of having a *finite* set of *rules* as well as a finite set of truths— 'basic concepts', *axioms* or whatever you call it—, so that you can then generate from them a (presumably infinite) set of *derived* truths. Nowadays we would describe the idea more simply, and say that Llull had just come across the idea of a *generative* system. In linguistics such a finitistic device is called a *grammar* ... and the generated strings are the *language*. In Computer Science the device is called a *machine* and what is being generated is the set of output configurations ... As is well known today, the same mechanism can run backwards: the same grammar that is capable of *generating* a language is also capable of *accepting* or *recognizing* its strings as belonging to it. Or the same machine which computes the batch of acceptable results is also capable of recognizing a correct calculation."

[167] Bonner (1997):14 (My emphasis).

[168] In a letter to Duke Ernst-August (Leibniz (1875–90):VII, 25.) In the opening paragraph of his "Historia et Origo Calculi Differentialis" (1714), he states: "It is an extremely useful thing to have knowledge of the true origins of memorable discoveries, especially those that have been found not by accident but by dint of meditation [i.e. deliberative reasoning]. It is not so much that thereby history may attribute to each man his own discoveries, … [but] that the art of making discoveries should be promoted by considering noteworthy examples of it." (Leibniz,1920:22)

[169] In my opinion one of the finest general introductions to Leibniz and his works is given in the 26 page introduction in Arthur (2014). The whole book is an excellent primer to Leibniz's life and work.

[170] Look (2021):3 "At present, the Leibniz-Archives in Hanover contain over 200,000 manuscript pages, including over 15,000 letters to his more than 1,000 correspondents, and 50,000 distinct essays, sketches, and exposés."

[171] Jaenecke (1996). "[Leibniz] tried to express knowledge in a system of symbols in such a way that reasoning could be performed by symbol manipulation. His plan [is] now called the LEIBNIZ project." (p. 88.) "… the Leibniz project is not a matter of logic but rather one of knowledge representation, a field largely unexploited in today's logic-oriented epistemology and philosophy of science." (Abstract). Cf. Finkelstein (1977); Rescher (1954); Mori (1996):482f.

[172] Some see it as beginning with the article by Jean van Heijenoort (1967): "Logic as Calculus and Logic as Language". However Lewis (1918) made the same distinction 50 years earlier:

> "Leibniz's conception of the task to be accomplished altered somewhat during his life, but two features characterize all the projects which he entertained: (1) a universal medium ("universal language" or "rational language" or "universal characteristic") for the expression of science; and (2) a calculus of reasoning (or "universal calculus") designed to display the most universal relations of scientific concepts and to afford some systematic abridgment of the labor of rational investigation in all fields, much as mathematical formulae abridge the labor of dealing with quantity and number. (Lewis 1918:6).

See also Rescher (1954:133), Hintikka (1997:***ix***); Peckhaus (2004); Lenzen (2018). Rohr (2021) makes the distinction in terms of "Logical Concepts vs. Logical Operations". Knobloch (2010: 289) expresses this in a slightly different way: "Leibniz spent his entire life trying to perfect and organize the *ars inveniendi*, the *ars characteristica*, and the *ars combinatoria*. These three arts were, in his view, inseparably connected." (Original emphasis.)

[173] Kneale & Kneale (1962):336f.

[174] See Benis-Sinaceur (1989/2023) for a detailed discussion.

[175] "On Universal Synthesis and Analysis, or the Art of Discovery and Judgment", (G., VII, 184–89). Cited in Leibniz (1969):223. Leibniz adds:

"… no one has attempted a language or characteristic [operative symbolic system] which includes at once both the arts of discovery and of judgment [*ars demonstrandi*], [and] one whose signs or characters serve the same purpose that arithmetical signs serve for numbers, and algebraic signs for quantities taken abstractly." Leibniz (1969):222. (Brackets added). Cf. Krämer (1988)

[176] Mugnai (2018):196. *Conjecture* – like *hypothesis* – renders a provisional (possible or plausible) explanation for a given situation or occurrence. Note also that *conjecture* was a central concept for the Neoplatonic theologian Nicolaus Cusanus (1401–1464) who greatly influenced Leibniz. In *De coniecturis* (c. 1441), Cusanus employed the notion in a way similar to that of Plato's "method of hypothesis", a precursor to Peirce's notion of abductive reasoning. It is also interesting to note that the term *conjecture* (con-jecture; con-jectus; con-iectura) is the Latin *calque* (i.e. a literal syllable by syllable translation) of the Greek term *sym-bolon* and *sym-ballo*. The etymological roots of both the Greek term and its Latin calque are the same: literally to "throw" or "put" things together: "a casting together" of e.g. facts or ideas. Synonyms are "compare", "**c**orrespond", "interpret" or hypothesise".

[177] Leibniz: (1923-):VI, 4. 913. Cited in Milkov (2006):9.

[178] Pombo (2010):2.

[179] *Ibid*. (Emphasis added).

[180] Cf. Uckelman (2010) for further discussion.

[181] *Analysis ad Alias Res quam Quantitates Applicata*, A VI 3 413-414, cited in Esquisabel (2008).

Leibniz also writes:

> "Calculus is nothing more … than an operation by means of characters [symbols], which takes place *not only in quantity*, but also in all other reasoning."... "[It] consists in the production of relations, which is accomplished by transformations of formulae, which in turn are carried out according to certain prescribed rules." *Fundamenta calculi ratiocinatoris*, (c. 1688); A VI 4, 921. Cited in Esquisabel (2012):23.

[182] Whitehead (1998):viii.

[183] See e.g. Rescher (1954); Lenzen (1989, 2004); Swoyer (1994); Zalta (2000); Milkov (2006); Levey (2011); Mugnai (2010, 2018, 2019); Malink & Vasudevan (2016).

[184] There have been two explanations (not mutually exclusive) for why Leibniz called this operation "real": 1) in order to distinguish this conceptual (non-mathematical) calculus from an arithmetic calculus, and thereby highlight the relationships between "real concepts" and not simply between mathematical entities, and 2) like the concept of a "real definition", it also emphasizes the *possibility* of the new concept being formed by the operation (i.e. its non-contradictory nature).

[185] Swoyer 1994:7. (Emphasis added). There is also a mereological parallel:

> "The notion of Real Addition is similar to that of mereological sum or fusion of contemporary mereology: the idea is that we can add or fuse different things and so obtain aggregates of those objects." (Costantini, 2021:273).

[186] Here I have reversed the order of the first two axioms for purely formal reasons. Note also that these three axioms are also principle axioms in lattice theory, which Leibniz also explored.

[187] Note that Leibniz did not explicitly formulate an associative "axiom", but nonetheless presumed it, since he applied it in postulates and within later operations.

[188] Generally, the *extension* of a concept consists of the class of things to which particular examples of the concept apply (i.e. its "members"). *Intension*, on the other hand, consists of the *properties* which the concept implies. Cf. Rescher, 1964:26. Mugnai (2018):194: "The operation of juxtaposition employed in the *General Inquiries* and the *real addition* of the essays on plus-minus calculus amount to the same operation performed by joining together two or more *concepts*". (Emphasis added.)

[189] Whitehead (1911):730.

[190] Leibniz, 1966:142. (Emphasis added.)

[191] See Rescher (2006):50f for further discussion.

[192] Leibniz (1923–):VI 4: 2228–40. For more details see Kim (2021).

[193] Leibniz (1923–):VI 4: 2231–2. Cited in Rescher (2006):50. [My brackets]. Note that, whether by lapse or typo, Leibniz (or a copier/translator) leaves out the tenth three letter possibility CDE, which is part of ACDE.

[194] The middle combinatoric connections of the "meet lattice" are left out to save space.

[195] DISSERTATIO De ARTE COMBINATORIA, In qua Ex Arithmeticae fundamentis Complicationum ac Transpositionum Doctrina novis praeceptis exstruitur, et usus ambarum per universum scientiarum orbem ostenditur; nova etiam Artis Meditandi, Seu Logicae Inventionis semina sparguntur.

I wish here to acknowledge the great service that Professor Massimo Mugnai and his colleagues, Han van Ruler and Martin Wilson, have performed in providing the research community with both the first full English translation of this work, and also an excellent introduction and analysis (Mugnai, et. al. (2020)). **Note**: At the time of this writing, a *first edition* copy of Leibniz's DISSERTATIO (1666) was available over the Internet for $185,000. One need only click on "Add to Cart".

[196] For a description of the software application, see Ritchey (2003).

[197] Note that there are terminological and notational differences between Aristotle's original version and the 17th century extended version. Couturat's notation is used here in both cases. (Couturat, 1901/2, Chapter 1: Syllogism.) For those interested in a more detailed account of Leibniz's approach to modelling the extended form of the syllogistic, see e.g. Magnai et. al. (2020). Cf. Robering (2014).

[198] Frisesmo (IEO):

Some A is B (**I**)	*Some mammals* can *fly* (**I**)
No C is A (**E**)	*No birds* are *mammals* (**E**)
∴ *Some not-C are B* (**O**)	∴ Some (creatures) that are *not birds* can *fly* (**O**)

[199] Taken from Robering (2014). Note that Leibniz (and Robering) use "J" instead of "I" for the Indefinite quantity.

[200] This short biography includes excerpts of text appearing on the site of the Swedish Morphological Society (from 2003), in Ritchey (2011) and also from a book review (Ritchey, 2012) of Alfred Stöckli & Roland Müller (2011) *Fritz Zwicky – An Extraordinary Astrophysicist*.

[201] See Rabouin (2020) for a detailed account of the history of the *Hannover manuscripts*.

[202] One can also mention the work of Peano, Frege and Russell which contributed to the "revival", but these were not predominantly engaged in publishing Leibniz's Hannover manuscripts.

[203] Couturat (1901): Chapter 6.

[204] See Spekkens (2019):6; cf. Heinzmann & Stump (2017).

[205] Poincaré (1921):386. Cf. Ippoliti (2018):13: "Poincaré draws on the idea of expressing and formalizing the unconscious processes that underlie the search for a solution to a problem. He argues that in order to solve problems our unconscious first combines and then selects ideas: when we face a problem and our conscious attempt to solve it does not succeed and stalls, the unconscious processes step in and work by creating all the possible combinations starting from certain ideas."

[206] Ostwald (1911):18. Cited in Hapke (2012):288

[207] The expression *General Morphological Analysis* was first used for Zwicky's modelling method in Ritchey (2006). Note also that applications of constraint-based combinatorial modelling are also found today in several other areas: e.g. in Information Science (IS), in what is called *topic analysis* and *topic modelling*; in *combinatorial optimization* modelling techniques; and in *formal concept analysis* in general. And, as mentions earlier (§1.3), the Artificial Intelligence community – which by its very nature has a deep interest in the epistemology of conceptual modelling – has developed its own notion of "constraint-based modelling" in what is called "constraint satisfaction problems" (CSP):

> "Since the early 1970s, researchers in artificial intelligence (AI) have investigated a class of combinatorial problems that became known as constraint-satisfaction problems (CSP).

> ... The study of constraint satisfaction occupies a prominent place in artificial intelligence, because many problems that arise in different areas can be modelled as constraint-satisfaction problems in a natural way...". Kolaitis & Vardi (2008):125 (emphasis added).

[208] Zwicky (1969):34. (Emphasis added)

[209] Cf. Rudnicki (1989).

[210] The former Zwicky residence in Varna is now a tourist attraction.

[211] The idea (on the basis of the general theory of relativity) that a star's gravitational field could produce a *lensing effect* on light was first theorized in 1924 by the Russian astronomer Orest Khvolson (1852–1934). In 1936 this effect was "quantified" by Albert Einstein, who showed that – because of the great distance to any nearby star – the "focus" of such a lensing effect would be far too short to be observable from earth. Zwicky did the mathematics and found that a galaxy would do the trick.

[212] Fritz Zwicky (1948):123.

[213] This section includes a review of text appearing in Ritchey (2011; 2013; 2018).

[214] Leibniz: A VI, 4. 913. Cited in Milkov (2006):9.

[215] Zwicky (1948):121f (italics added).

[216] Zwicky (1960):1.

[217] Rittel (1973). The term "wicked problem" was apparently prompted by a remark by Churchman (1967).

[218] See Conklin (2006) for a more detailed account.

[219] The "Criteria quotes" and their numbering are from Rittel & Webber (1973).

[220] Zwicky (1960), pp.118 & 164.

[221] From Morphological Astronomy (1948).

[222] Zwicky's publications are listed on the site of the Fritz Zwicky Stiftung, at: www.zwicky-stiftung.ch.

[223] For an overview of publications in these different areas see Álvarez & Ritchey (2015).

[224] At that time this was a division of the Swedish Defence Research Agency (FOA).

[225] A description of the MA/Carma software is available at https://www.swemorph.com/macarma.html